

Non-analytic Superposition Results on Modulation Spaces with Subexponential Weights

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Abstract. Motivated by classical results for Gevrey spaces and their applications to nonlinear partial differential equations we define so-called Gevrey-modulation spaces. We establish analytic as well as non-analytic superposition results on Gevrey-modulation spaces. These results are extended to a special weighted modulation space where the weight increases stronger than any polynomial but less than as in the Gevrey case.

Keywords. modulation spaces, subexponential weights, Gevrey spaces, multiplication algebras, superposition operators.

1. Introduction

Gevrey analysis is an effective tool to treat several models of partial differential equations. Instead of treating the model in the physical space considerations in the phase space are more appropriate. For us, a Gevrey function can be characterized by it's behavior on the Fourier transform side, i.e.,

$$f \in \mathcal{G}_s \iff e^{\langle \xi \rangle^{\frac{1}{s}}} \mathcal{F}f(\xi) \in L^2(\mathbb{R}^n),$$

where $s > 1$.

In [4] the Gevrey example is considered, that is, the Cauchy problem

$$u_{tt} - u_x = 0, \quad u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x). \quad (1.1)$$

It is globally (in time) well-posed in the Gevrey space \mathcal{G}_s if and only if $s < 2$. Another Cauchy problem which is reasonable to consider in Gevrey spaces is the following one:

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x). \quad (1.2)$$

If we suppose that the positive coefficient $a = a(t)$ belongs to the Hölder space $C^\alpha[0, T]$, $0 < \alpha < 1$, then (1.2) is globally (in time) well-posed in the Gevrey space \mathcal{G}_s for $s < \frac{1}{1-\alpha}$ which is explained in [6].

If one is interested in the solvability behavior of the corresponding semi-linear Cauchy problems

$$u_{tt} - u_x = f(u), \quad u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x), \quad (1.3)$$

or

$$u_{tt} - a(t)u_{xx} = f(u), \quad u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x) \quad (1.4)$$

with an admissible nonlinearity $f = f(u)$ and Gevrey data ϕ and ψ , then one of the first steps is to explain superposition operators in \mathcal{G}_s . This was done in [4]. There appropriate superposition operators are studied in spaces which are defined by the behavior of the Fourier transform as in case of \mathcal{G}_s .

In the present paper we devote ourselves to modulation spaces. In various papers Wang et al [30, 29, 27, 28] have shown that modulation spaces may serve as a reasonable tool when studying existence and regularity of linear and nonlinear partial differential equations. It will turn out that modulation spaces equipped with some admissible subexponential weights allow to prove for superposition operators similar results as in [4].

The paper is organized as follows. First of all an approach to modulation spaces $M_{p,q}^s(\mathbb{R}^n)$ and Gevrey-modulation spaces $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is chosen as introduced in [30]. After obtaining boundedness of functions in Gevrey-modulation spaces we are interested in investigating the behavior of analytic nonlinearities. Therefore it is sufficient to prove algebra properties which is done in Section 2.

In Section 3 we are able to find non-analytic functions $f \in C^\infty(\mathbb{R}^n)$ such that the corresponding superposition operator T_f defined by

$$T_f : u \rightarrow T_f u := f(u)$$

maps the Gevrey-modulation space $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ into itself.

In Section 4 we shall study superposition operators in a special modulation space of ultra-differentiable functions with another type of subexponential weight. This space contains all Gevrey spaces.

The main concern of this paper is to prove some analytic as well as non-analytic superposition results in particularly weighted modulation spaces.

Some open problems and concluding remarks complete the paper (see Section 5).

2. Modulation Spaces

2.1. Definitions

First of all we introduce some basic notations and definitions. In \mathbb{R}^n the notation of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ is used, where $|\alpha| = \sum_{j=1}^n \alpha_j$. Given two multi-indices α and β , then $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for $1 \leq j \leq n$. Furthermore let f be a function on \mathbb{R}^n and $x \in \mathbb{R}^n$, then

$$x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$$

and

$$D^\alpha f(x) = \frac{1}{1^{|\alpha|}} \frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{1}{1^{|\alpha|}} \left(\prod_{j=1}^n \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} \right) f(x).$$

A function $f \in C^\infty(\mathbb{R}^n)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ if and only if

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty$$

for all multi-indices α, β . The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$ which is the dual space of $\mathcal{S}(\mathbb{R}^n)$.

We introduce $\langle \xi \rangle_m^s := (m^2 + |\xi|^2)^{\frac{s}{2}}$. If $m = 1$, then we write $\langle \xi \rangle^s$ for simplicity. The notation $a \lesssim b$ is equivalent to $a \leq Cb$ with a positive constant C . Moreover, by writing $a \asymp b$ we mean $a \lesssim b \lesssim a$. The Fourier transform of an admissible function f is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx \quad (x, \xi \in \mathbb{R}^n).$$

Analogously the inverse Fourier transform is defined by

$$\mathcal{F}^{-1} \hat{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} d\xi \quad (x, \xi \in \mathbb{R}^n).$$

In order to describe local frequency properties of a function f we define the following joint time-frequency representation.

Definition 2.1. Let ϕ be the so-called window function which is a fixed function that is not identically zero. Then the short-time Fourier transform (STFT) of a function f with respect to ϕ is defined as

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(s) \overline{\phi(s - x)} e^{-is \cdot \xi} ds \quad (x, \xi \in \mathbb{R}^n).$$

We want to define a family of Banach spaces which controls globally the joint time-frequency information. Therefore we introduce weighted modulation spaces, where we will use particular weights. A more detailed discussion about weight functions can be found in Chapter 11 in [10].

Definition 2.2. Let $1 \leq p, q \leq \infty$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed window and assume $s, \sigma \in \mathbb{R}$ to be the weight parameters. Then the weighted modulation space $\dot{M}_{p,q}^{s,\sigma}(\mathbb{R}^n)$ is the set

$$\dot{M}_{p,q}^{s,\sigma}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{M}_{p,q}^{s,\sigma}} < \infty\},$$

where the norm is defined as

$$\|f\|_{\dot{M}_{p,q}^{s,\sigma}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\phi f(x, \xi) \langle x \rangle^\sigma \langle \xi \rangle^s|^p dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}}.$$

For $p = \infty$ and/or $q = \infty$ the definition can be obviously modified by taking L^∞ -norms.

Remark 2.3. (i) If $s = \sigma = 0$, then we obtain the so-called standard modulation space $\dot{M}_{p,q}(\mathbb{R}^n)$. For $\sigma = 0$, i.e., no weight with respect to the x -variable, the weighted modulation space is denoted by $\dot{M}_{p,q}^s(\mathbb{R}^n)$. Subsequently the space $\dot{M}_{p,q}^{s,\sigma}(\mathbb{R}^n)$ is just referred to as modulation space unless it is explicitly stated differently.

(ii) A rough interpretation is as follows. The weight in x in the preceding definition corresponds to some growth or decay properties of f . On the other hand the weight in ξ corresponds to regularity properties of f in $\dot{M}_{p,q}^{s,\sigma}(\mathbb{R}^n)$.

(iii) General references with respect to (weighted) modulation spaces are Feichtinger [7], Grochenig [10], Toft [24], Triebel [26] and Wang et. al [30] to mention only a few.

At this point we want to go back to the alternative approach to the STFT. Since we aim at specific superposition results on weighted modulation spaces it will turn out that introducing the following approach to the STFT will be convenient. We are basically adopting the idea of obtaining local frequency properties of a function f . Related frequency decomposition techniques are explained in [10]. A special case, the so-called frequency-uniform decomposition, was independently introduced by Wang (e.g., see [29]). Let $\rho : \mathbb{R}^n \mapsto [0, 1]$ be a Schwartz function which is compactly supported in the cube

$$Q_0 := \{\xi \in \mathbb{R}^n : -1 \leq \xi_i \leq 1, i = 1, \dots, n\}.$$

Moreover,

$$\rho(\xi) = 1 \quad \text{if} \quad |\xi_i| \leq \frac{1}{2}, \quad i = 1, 2, \dots, n.$$

With $\rho_k(\xi) := \rho(\xi - k)$, $\xi \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$, it follows

$$\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1 \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

Finally we define

$$\sigma_k(\xi) := \rho_k(\xi) \left(\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad \xi \in \mathbb{R}^n, \quad k \in \mathbb{Z}^n.$$

The following properties are obvious:

- $0 \leq \sigma_k(\xi) \leq 1$ for all $\xi \in \mathbb{R}^n$;
- $\text{supp } \sigma_k \subset Q_k := \{\xi \in \mathbb{R}^n : -1 \leq \xi_i - k_i \leq 1, i = 1, \dots, n\}$;
- $\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1$ for all $\xi \in \mathbb{R}^n$;
- There exists a constant $C > 0$ such that $\sigma_k(\xi) \geq C$ if $\max_{i=1, \dots, n} |\xi_i - k_i| \leq \frac{1}{2}$;
- For all $m \in \mathbb{N}_0$ there exist positive constants C_m such that for $|\alpha| \leq m$

$$\sup_{k \in \mathbb{Z}^n} \sup_{\xi \in \mathbb{R}^n} |D^\alpha \sigma_k(\xi)| \leq C_m.$$

The operator

$$\square_k := \mathcal{F}^{-1}(\sigma_k \mathcal{F}(\cdot)), \quad k \in \mathbb{Z}^n,$$

is called uniform decomposition operator.

As it is well-known there is an equivalent description of the modulation spaces by means of the uniform decomposition operator, see Feichtinger [7].

Definition 2.4. Let $1 \leq p, q \leq \infty$ and assume $s \in \mathbb{R}$ to be the weight parameter. Suppose the window $\rho \in \mathcal{S}(\mathbb{R}^n)$ is compactly supported. Then the weighted modulation space $M_{p,q}^s(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that their norm

$$\|f\|_{M_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_{L^p}^q \right)^{\frac{1}{q}}$$

is finite with obvious modifications when $p = \infty$ and/or $q = \infty$.

Proposition 2.5. *The norms of the Definitions 2.2 and 2.4 are equivalent. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then it holds*

$$C_1 \|f\|_{\dot{M}_{p,q}^s} \leq \|f\|_{M_{p,q}^s} \leq C_2 \|f\|_{\dot{M}_{p,q}^s},$$

where the positive constants C_1 and C_2 are depending on the dimension n , the window function and on the frequency-uniform decomposition, respectively.

Proof. Cf. Proposition 2.1 in [29] and Proposition 1.12 in [16]. \square

In what follows we shall always work with the uniform decomposition operator.

2.2. Gevrey-Modulation Spaces

There is a famous classical result by Katznelson [13] (in the periodic case) and by Helson, Kahane, Katznelson, Rudin [11] (nonperiodic case) which says that only analytic functions operate on the Wiener algebra $\mathcal{A}(\mathbb{R}^n)$. More exactly, the operator $T_f : u \mapsto f(u)$ maps $\mathcal{A}(\mathbb{R}^n)$ into $\mathcal{A}(\mathbb{R}^n)$ if and only if $f(0) = 0$ and f is analytic. Here $\mathcal{A}(\mathbb{R}^n)$ is the collection of all $u \in C(\mathbb{R}^n)$ such that $\mathcal{F}u \in L^1(\mathbb{R}^n)$. Moreover, a similar result is obtained for particular standard modulation spaces. In [1] it is stated that T_f maps $M_{1,1}$ into $M_{1,1}$ if and only if $f(0) = 0$ and f is analytic. Therefore, the existence of non-analytic superposition results for weighted modulation spaces is a priori not so clear. Here we are interested in weighted modulation spaces with different weights than used above.

Definition 2.6. The integrability parameters are given by $1 \leq p, q \leq \infty$. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a fixed window and assume $s > 0$ to be the weight parameter. By $(\sigma_k)_k$ we denote the associated uniform decomposition of unity in the above sense. Then the Gevrey-modulation space $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is the collection of all $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{GM}_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} e^{q|k|^{\frac{1}{s}}} \|\square_k f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty$$

(with obvious modifications when $p = \infty$ and/or $q = \infty$).

Remark 2.7. (i) Modulation spaces with general weights have been also considered by Gol'dman [9] (general function spaces of Besov type) and Triebel [26] (trace problems).

(ii) We shall call the weights $w(x) := e^{|x|^{\frac{1}{s}}}$, $x \in \mathbb{R}^n$, with $s > 1$ subexponential.

It is not difficult to prove the following basic facts.

Lemma 2.8. (i) *The Gevrey-modulation space $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is a Banach space.*
(ii) *$\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is independent of the choice of the window $\rho \in C_0^\infty(\mathbb{R}^n)$ in the sense of equivalent norms.*
(iii) *$\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ has the Fatou property, i.e., if $(f_m)_{m=1}^\infty \subset \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is a sequence such that $f_m \rightharpoonup f$ (weak convergence in $S'(\mathbb{R}^n)$) and*

$$\sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{GM}_{p,q}^s} < \infty,$$

then $f \in \mathcal{GM}_{p,q}^s$ follows and

$$\|f\|_{\mathcal{GM}_{p,q}^s} \leq \sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{GM}_{p,q}^s} < \infty,$$

Proof. It is enough to comment on a proof of (iii). We follow [8]. From assumption, it follows that for all $k \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$,

$$\mathcal{F}^{-1}[\sigma_k \mathcal{F} f_m](x) = (2\pi)^{-n/2} f_m(x - \cdot)(\sigma_k) \rightarrow f(x - \cdot)(\sigma_k) = \mathcal{F}^{-1}[\sigma_k \mathcal{F} f](x)$$

as $m \rightarrow \infty$. Fatou's lemma yields

$$\sum_{|k| \leq N} \left(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\sigma_k \mathcal{F} f](x)|^p dx \right)^{\frac{q}{p}} \leq \liminf_{m \rightarrow \infty} \sum_{|k| \leq N} \left(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\sigma_k \mathcal{F} f_m](x)|^p dx \right)^{\frac{q}{p}}.$$

An obvious monotonicity argument completes the proof. \square

Obviously the spaces $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ are monotone in s and q . But they are also monotone with respect to p . To show this we recall Nikol'skij's inequality, see, e.g., Nikol'skij [15, 3.4] or Triebel [25, 1.3.2].

Lemma 2.9. *Let $1 \leq p \leq q \leq \infty$ and f be an integrable function with $\text{supp } \mathcal{F}f(\xi) \subset B(y, r)$, i.e., the support of the Fourier transform of f is contained in a ball with radius $r > 0$ and center in $y \in \mathbb{R}^n$. Then it holds*

$$\|f\|_{L^q} \leq C r^{n(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p}$$

with a constant $C > 0$ independent of r and y .

This implies $\|\square_k f\|_{L^q} \leq c \|\square_k f\|_{L^p}$ if $p \leq q$ with c independent of k and f which results in the following corollary.

Corollary 2.10. *Let $0 < s_0 < s$, $p_0 < p$ and $q_0 < q$. Then the following embeddings hold and are continuous:*

$$\mathcal{M}_{p,q}^{s_0}(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{p,q}^s(\mathbb{R}^n), \quad \mathcal{GM}_{p_0,q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$$

and

$$\mathcal{GM}_{p,q_0}^s(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{p,q}^s(\mathbb{R}^n);$$

i.e., for all p, q , $1 \leq p, q \leq \infty$ we have

$$\mathcal{GM}_{1,1}^s(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{p,q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{\infty,\infty}^s(\mathbb{R}^n).$$

Only very smooth functions have a chance to belong to one of the spaces $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$.

Corollary 2.11. *Let $s > 0$ and $1 \leq p, q \leq \infty$. If $f \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$, then f is infinitely often differentiable and there exists a constant $C = C(f, n)$ such that*

$$|D^\alpha f(x)| \leq C^{|\alpha|+1} (\alpha!)^{sn}, \quad x \in \mathbb{R}^n. \quad (2.1)$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\varphi : \mathbb{R}^n \rightarrow [0, 1]$, $\varphi(\xi) = 1$ if $\xi \in \text{supp } \rho$ and $\text{supp } \varphi \subset \max_{i=1, \dots, n} |\xi_i| \leq 2$. We put $\varphi_k(\xi) := \varphi(\xi - k)$, $\xi \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$. In what follows we work with the distributional derivative, i.e., the assumption $f \in S'(\mathbb{R}^n)$ is sufficient. Hence, if $m > n/2$, there exists a constant c_1 such that

$$|\square_k(D^\alpha f)(x)| = |\mathcal{F}^{-1}[\varphi_k(\xi) \xi^\alpha \sigma_k(\xi) \mathcal{F}f(\xi)](x)| \leq c_1 \|\varphi_k(\xi) \xi^\alpha\|_{W_2^m} |\square_k f(x)|$$

holds for all f and all x , see, e.g., [21, Prop. 1.7.5]. By using

$$\begin{aligned} \|\varphi_k(\xi) \xi^\alpha\|_{W_2^m} &\leq |\text{supp } \varphi|^{1/2} \max_{|\gamma| \leq m} \sup_{\xi \in \text{supp } \varphi_k} |D^\gamma(\varphi_k(\xi) \xi^\alpha)| \\ &\leq c_2 (1 + |k|)^{|\alpha|} \end{aligned}$$

with c_2 independent of k and α we conclude

$$|\square_k(D^\alpha f)(x)| \leq c_3 (1 + |k|)^{|\alpha|} |\square_k f(x)|, \quad x \in \mathbb{R}^n.$$

Hence, for any N we have

$$\begin{aligned} \sum_{|k| < N} |\square_k(D^\alpha f)(x)| &\leq c_3 \sum_{|k| < N} (1 + |k|)^{|\alpha|} \|\square_k f(x)\|_{L^\infty} \\ &\leq c_3 \left(\sum_{|k| < N} (1 + |k|)^{|\alpha|} e^{-|k|^{1/s}} \right) \|f\|_{\mathcal{GM}_{\infty,\infty}^s} \\ &\leq c_4 \|f\|_{\mathcal{GM}_{\infty,\infty}^s}, \quad x \in \mathbb{R}^n, \end{aligned} \quad (2.2)$$

where c_4 does not depend on f, N and x . This implies convergence of $\left(\sum_{|k| < N} \square_k(D^\alpha f) \right)_N$ in $C_{ub}(\mathbb{R}^n)$. Since

$$\lim_{N \rightarrow \infty} \sum_{|k| < N} \square_k(D^\alpha f) = D^\alpha f \quad \text{in } S'(\mathbb{R}^n)$$

we conclude $D^\alpha f \in C_{ub}(\mathbb{R}^n)$ and

$$|D^\alpha f(x)| \leq \sum_{k \in \mathbb{Z}^n} |\square_k(D^\alpha f)(x)| \leq c_4 \|f\|_{\mathcal{GM}_{\infty,\infty}^s}, \quad x \in \mathbb{R}^n.$$

This proves $f \in C^\infty(\mathbb{R}^n)$. Now we turn back to the estimate (2.2). Observe

$$\begin{aligned} \sum_{|k| < N} (1 + |k|)^{|\alpha|} e^{-|k|^{1/s}} &\asymp \int_{\mathbb{R}^n} (1 + |x|)^{|\alpha|} e^{-|x|^{1/s}} dx \\ &= c_n \int_0^\infty (1 + r)^{|\alpha|+n-1} e^{-r^{1/s}} dr \\ &\asymp \int_0^\infty t^{s(|\alpha|+n)-1} e^{-t} dt = \Gamma(s(|\alpha| + n)), \end{aligned}$$

where the constants behind \asymp are independent of α . Next we apply $\Gamma(x) \leq x^{x-1}$. This yields

$$|D^\alpha f(x)| \leq c_5 (s(|\alpha| + n))^{(s(|\alpha|+n))-1} \|f\|_{\mathcal{GM}_{\infty,\infty}^s}.$$

Recall the multinomial theorem

$$(\alpha_1 + \dots + \alpha_n)^k = \sum_{|\beta|=k} \binom{k}{\beta} \alpha^\beta$$

and take into account that

$$\max_{|\beta|=|\alpha|} \alpha^\beta = \alpha^\alpha$$

we obtain with $k = |\alpha| + 1$

$$\begin{aligned} (s(|\alpha| + n))^{(s(|\alpha|+n))} &\leq (sn)^{sn} (sn)^{|\alpha|} \left((|\alpha| + 1)^{|\alpha|+1} \right)^{sn} \\ &\leq (sn)^{sn} (sn)^{|\alpha|} \left(\gamma^\gamma \sum_{|\beta|=k} \binom{k}{\beta} \right)^{sn} \\ &\leq (sn)^{sn} (sn)^{|\alpha|} n^{(|\alpha|+1)sn} (\gamma^\gamma)^{sn}. \end{aligned}$$

Here $\gamma = \alpha + e^j$, where $e^j = (0, \dots, 0, 1, 0, \dots, n)$ for some $j \in \{1, 2, \dots, n\}$. In case $\alpha_1 \neq 0$ we find

$$\begin{aligned} (s(|\alpha| + n))^{(s(|\alpha|+n))} &\leq (sn)^{sn} (sn)^{|\alpha|} n^{(|\alpha|+1)sn} \left((\alpha_1 + 1)^{\alpha_1+1} \prod_{j=2}^n \alpha_j^{\alpha_j} \right)^{sn} \\ &\leq (sn)^{sn} (sn)^{|\alpha|} (2n)^{(|\alpha|+1)sn} \left(\alpha_1 \prod_{j=1}^n \alpha_j^{\alpha_j} \right)^{sn} \\ &\leq (sn)^{sn} (sn)^{|\alpha|} (2n)^{(|\alpha|+1)sn} |\alpha|^{sn} (\alpha^\alpha)^{sn}. \end{aligned}$$

In case $\alpha_1 = 0$ we have to modify this argument in an obvious way. Inserting the obtained estimate in our previously found inequality we have proved

$$|D^\alpha f(x)| \leq c_5 (sn)^{sn} (sn)^{|\alpha|} (2n)^{(|\alpha|+1)sn} |\alpha|^{sn} (\alpha^\alpha)^{sn} \|f\|_{\mathcal{GM}_{\infty,\infty}^s}.$$

An application of Stirling's formula yields the claim. \square

Remark 2.12. (i) Inequality (2.1) makes clear that the elements of the spaces $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ have some classical Gevrey regularity, see, e.g., Rodino [17, Definition 1.4.1]. In fact, we proved $\mathcal{GM}_{p,q}^s(\mathbb{R}^n) \subset G^{sn}(\mathbb{R}^n)$.

(ii) The Corollary 2.11 remains true under the weaker assumption $f \in S'(\mathbb{R}^n)$

and $\|f\|_{\mathcal{GM}_{\infty,\infty}^s} < \infty$. As a consequence we observe that a replacement of the requirement $f \in L^p(\mathbb{R}^n)$ by $f \in S'(\mathbb{R}^n)$ in Definition 2.6 does not change the space $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$.

For $p = q = 2$ we can simplify the description of $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$.

Lemma 2.13. A tempered distribution f belongs to $\mathcal{GM}_{2,2}^s(\mathbb{R}^n)$ if and only if $e^{|\cdot|^{1/s}} \mathcal{F}f(\cdot) \in L^2(\mathbb{R}^n)$.

Proof. This follows from

$$\begin{aligned} \|f\|_{\mathcal{GM}_{2,2}^s}^2 &= \sum_{k \in \mathbb{Z}^n} e^{2|k|^{\frac{1}{s}}} \|\square_k f\|_{L^2}^2 \\ &= \sum_{k \in \mathbb{Z}^n} e^{2|k|^{\frac{1}{s}}} \int |\sigma_k(\xi) \mathcal{F}f(\xi)|^2 d\xi \\ &\asymp \sum_{k \in \mathbb{Z}^n} e^{2|k|^{\frac{1}{s}}} \int_{\|\xi-k\|_\infty \leq 1} |\mathcal{F}f(\xi)|^2 d\xi \\ &\asymp \sum_{k \in \mathbb{Z}^n} \int_{\|\xi-k\|_\infty \leq 1} e^{2|\xi|^{\frac{1}{s}}} |\mathcal{F}f(\xi)|^2 d\xi \asymp \int_{\mathbb{R}^n} e^{2|\xi|^{\frac{1}{s}}} |\mathcal{F}f(\xi)|^2 d\xi, \end{aligned}$$

where we used the properties of our decomposition of unity. □

2.3. Multiplication Algebras

In the next step we want to prove an essential property for Gevrey-modulation spaces. Subsequently we always mean algebras under pointwise multiplication when speaking of algebras. This property is important in two ways. On the one hand it is needed to handle semi-linear problems as (1.3) or (1.4) with analytic nonlinearity $f(u)$. On the other hand it is a useful tool in the proof of the non-analytic superposition result in Section 3. At this point we can already mention results of Iwabuchi in [12]. However he imposed particular conditions on the integrability parameters. Of some importance for our proof will be the following elementary lemma, see [2], [4].

Lemma 2.14. Let $s > 1$ and put $\delta := 2 - 2^{1/s}$. Then

$$e^{|k|^{\frac{1}{s}}} \leq e^{|l|^{\frac{1}{s}}} e^{|l-k|^{\frac{1}{s}}} e^{-\delta \min\{|l-k|, |l|\}^{\frac{1}{s}}},$$

holds for arbitrary $k, l \in \mathbb{Z}^n$.

After these preparations we can state the main result of this section.

Theorem 2.15. Let $1 \leq p_1, p_2, q \leq \infty$ and $s > 1$. Define p by $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$ and assume that $f \in \mathcal{GM}_{p_1,q}^s(\mathbb{R}^n)$ and $g \in \mathcal{GM}_{p_2,q}^s(\mathbb{R}^n)$. Then $f \cdot g \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ and it holds

$$\|f \cdot g\|_{\mathcal{GM}_{p,q}^s} \leq C \|f\|_{\mathcal{GM}_{p_1,q}^s} \|g\|_{\mathcal{GM}_{p_2,q}^s}$$

with a positive constant C which only depends on the choice of the frequency-uniform decomposition, the dimension n and the parameters s and q .

Proof. Later on we shall use the same strategy of proof as below in slightly different situations. For this reason and later use we shall take care of all constants showing up in our estimates below.

We know that $\text{supp } \sigma_k \subset Q_k := \{\xi \in \mathbb{R}^n : -1 \leq \xi_i - k_i \leq 1, i = 1, \dots, n\}$. Further on we introduce the notations $f_j(x) = \mathcal{F}^{-1}(\sigma_j \mathcal{F}f)(x)$ and $g_l(x) = \mathcal{F}^{-1}(\sigma_l \mathcal{F}g)(x)$ for $j, l \in \mathbb{Z}^n$. At least formally we have the following representation of the product $f \cdot g$ as

$$f \cdot g = \sum_{j, l \in \mathbb{Z}^n} f_j \cdot g_l.$$

Hölder's inequality yields

$$\begin{aligned} \left| \sum_{j, l \in \mathbb{Z}^n} f_j \cdot g_l \right| &\leq \left(\sum_{j \in \mathbb{Z}^n} \|f_j\|_{L^\infty}^2 \right)^{1/2} \left(\sum_{l \in \mathbb{Z}^n} \|g_l\|_{L^\infty}^2 \right)^{1/2} \\ &\leq C \|f\|_{\mathcal{GM}_{\infty, \infty}^s} \|g\|_{\mathcal{GM}_{\infty, \infty}^s} \end{aligned}$$

for any $s > 0$ and with a constant C independent of f and g . This shows convergence of $\sum_{j, l \in \mathbb{Z}^n} f_j \cdot g_l$ in $C_{ub}(\mathbb{R}^n)$, hence in $S'(\mathbb{R}^n)$. In view of Lemma 2.8(iii) it will be sufficient to prove that the sequence $(\sum_{|k|, |l| < N} f_j g_l)_N$ is uniformly bounded in $\mathcal{GM}_{p, q}^s(\mathbb{R}^n)$.

Determining the Fourier support of $f_j \cdot g_l$ we see that

$$\begin{aligned} \text{supp } \mathcal{F}(f_j g_l) &= \text{supp } (\mathcal{F}f_j * \mathcal{F}g_l) \\ &\subset \{\xi \in \mathbb{R}^n : j_i + l_i - 2 \leq \xi_i \leq j_i + l_i + 2, i = 1, \dots, n\}. \end{aligned}$$

Hence, the term $\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_j \cdot g_l))$ vanishes if $\|k - (j + l)\|_\infty \geq 3$. So we obtain

$$\begin{aligned} \sigma_k \mathcal{F}(f \cdot g) &= \sigma_k \mathcal{F}\left(\sum_{j, l \in \mathbb{Z}^n} f_j \cdot g_l\right) = \sigma_k \mathcal{F}\left(\sum_{\substack{j, l \in \mathbb{Z}^n, \\ k_i - 3 < j_i + l_i < k_i + 3, \\ i = 1, \dots, n}} f_j g_l\right) \\ &\stackrel{[r=j+l]}{=} \sum_{\substack{r \in \mathbb{Z}^n, \\ k_i - 3 < r_i < k_i + 3, \\ i = 1, \dots, n}} \sum_{l \in \mathbb{Z}^n} \sigma_k \mathcal{F}(f_{r-l} g_l). \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f \cdot g))\|_{L^p} &\leq \sum_{\substack{r \in \mathbb{Z}^n, \\ k_i - 3 < r_i < k_i + 3, \\ i = 1, \dots, n}} \sum_{l \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_{r-l} g_l))\|_{L^p} \\ &\stackrel{[t=r-k]}{=} \sum_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i = 1, \dots, n}} \sum_{l \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_{t-(l-k)} g_l))\|_{L^p}. \end{aligned}$$

These preparations yield the following norm estimates

$$\begin{aligned}
& \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f \cdot g))\|_{L^p}^q \right)^{\frac{1}{q}} \\
& \leq \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \left[\sum_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \sum_{l \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_{t-(l-k)} g_l))\|_{L^p} \right]^q \right)^{\frac{1}{q}} \\
& \leq \sum_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \left[\sum_{l \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_{t-(l-k)} g_l))\|_{L^p} \right]^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Observe

$$\begin{aligned}
\|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f_{t-(l-k)} g_l))\|_{L^p} &= (2\pi)^{-n/2} \|(\mathcal{F}^{-1} \sigma_k) * (f_{t-(l-k)} g_l)\|_{L^p} \\
&\leq (2\pi)^{-n/2} \|\mathcal{F}^{-1} \sigma_k\|_{L^1} \|f_{t-(l-k)} g_l\|_{L^p} \\
&= (2\pi)^{-n/2} \|\mathcal{F}^{-1} \sigma_0\|_{L^1} \|f_{t-(l-k)} g_l\|_{L^p},
\end{aligned}$$

where we used Young's inequality. We put $c_1 := (2\pi)^{-n/2} \|\mathcal{F}^{-1} \sigma_0\|_{L^1}$. This implies

$$\begin{aligned}
& \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f \cdot g))\|_{L^p}^q \right)^{\frac{1}{q}} \\
& \leq c_1 \sum_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \left[\sum_{l \in \mathbb{Z}^n} \|f_{t-(l-k)} g_l\|_{L^p} \right]^q \right)^{\frac{1}{q}}.
\end{aligned}$$

We continue by using Hölder's inequality to get

$$\begin{aligned}
& \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f \cdot g))\|_{L^p}^q \right)^{\frac{1}{q}} \\
& \leq c_2 \max_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \left[\sum_{l \in \mathbb{Z}^n} \|f_{t-(l-k)}\|_{L^{p_1}} \|g_l\|_{L^{p_2}} \right]^q \right)^{\frac{1}{q}} \quad (2.3)
\end{aligned}$$

with $c_2 := c_1 5^n$. Lemma 2.14 yields

$$\begin{aligned}
e^{|k|^{\frac{1}{s}}} & \left[\sum_{l \in \mathbb{Z}^n} \|f_{t-(l-k)}\|_{L^{p_1}} \|g_l\|_{L^{p_2}} \right] \\
& \leq \sum_{\substack{l \in \mathbb{Z}^n, \\ |l| \leq |l-k|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta|l|^{\frac{1}{s}}} \\
& \quad + \sum_{\substack{l \in \mathbb{Z}^n, \\ |l-k| \leq |l|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta|l-k|^{\frac{1}{s}}}.
\end{aligned}$$

Both parts of this right-hand side will be estimated separately. We put

$$\begin{aligned}
S_{1,t,k} &:= \sum_{\substack{l \in \mathbb{Z}^n, \\ |l| \leq |l-k|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta|l|^{\frac{1}{s}}}, \\
S_{2,t,k} &:= \sum_{\substack{l \in \mathbb{Z}^n, \\ |l-k| \leq |l|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta|l-k|^{\frac{1}{s}}}.
\end{aligned}$$

With $\frac{1}{q} + \frac{1}{q'} = 1$ we find

$$\begin{aligned}
S_{1,t,k} & \stackrel{[j=l-k]}{=} \sum_{\substack{j \in \mathbb{Z}^n, \\ |j+k| \leq |j|}} e^{|j|^{\frac{1}{s}}} \|f_{t-j}\|_{L^{p_1}} e^{|j+k|^{\frac{1}{s}}} \|g_{j+k}\|_{L^{p_2}} e^{-\delta|j+k|^{\frac{1}{s}}} \\
& \leq \left(\sum_{\substack{j \in \mathbb{Z}^n, \\ |j+k| \leq |j|}} \left| e^{|j|^{\frac{1}{s}}} \|f_{t-j}\|_{L^{p_1}} e^{|j+k|^{\frac{1}{s}}} \|g_{j+k}\|_{L^{p_2}} \right|^q \right)^{\frac{1}{q}} \\
& \quad \times \left(\sum_{\substack{j \in \mathbb{Z}^n, \\ |j+k| \leq |j|}} \left| e^{-\delta|j+k|^{\frac{1}{s}}} \right|^{q'} \right)^{\frac{1}{q'}}.
\end{aligned}$$

Since

$$\left(\sum_{\substack{j \in \mathbb{Z}^n, \\ |j+k| \leq |j|}} \left| e^{-\delta|j+k|^{\frac{1}{s}}} \right|^{q'} \right)^{\frac{1}{q'}} \leq \left(\sum_{m \in \mathbb{Z}^n} e^{-\delta|m|^{\frac{1}{s} q'}} \right)^{\frac{1}{q'}} =: c_3 \quad (2.4)$$

we conclude

$$\begin{aligned}
\left(\sum_{k \in \mathbb{Z}^n} S_{1,t,k}^q \right)^{1/q} & \leq c_3 \left(\sum_{k \in \mathbb{Z}^n} \sum_{\substack{j \in \mathbb{Z}^n, \\ |j+k| \leq |j|}} e^{|j|^{\frac{1}{s} q}} \|f_{t-j}\|_{L^{p_1}}^q e^{|j+k|^{\frac{1}{s} q}} \|g_{j+k}\|_{L^{p_2}}^q \right)^{1/q} \\
& \leq c_3 \left(\sum_{j \in \mathbb{Z}^n} e^{|j|^{\frac{1}{s} q}} \|f_{t-j}\|_{L^{p_1}}^q \sum_{k \in \mathbb{Z}^n} e^{|j+k|^{\frac{1}{s} q}} \|g_{j+k}\|_{L^{p_2}}^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Because of $|j|^{1/s} \leq |j - t|^{1/s} + |t|^{1/s}$ we know

$$\max_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \sup_{j \in \mathbb{Z}^n} e^{|j|^{1/s} - |t-j|^{1/s}} \leq \max_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} e^{|t|^{1/s}} \leq e^{(2\sqrt{n})^{1/s}} =: c_4 < \infty. \quad (2.5)$$

This implies

$$\left(\sum_{k \in \mathbb{Z}^n} S_{1,t,k}^q \right)^{1/q} \leq c_3 c_4 \|g\|_{\mathcal{GM}_{p_2,q}^s} \|f\|_{\mathcal{GM}_{p_1,q}^s},$$

where c_3, c_4 are independent of f, g and t . For the second sum the estimate

$$\left(\sum_{k \in \mathbb{Z}^n} S_{2,t,k}^q \right)^{1/q} \leq c_5 \|g\|_{\mathcal{GM}_{p_2,q}^s} \|f\|_{\mathcal{GM}_{p_1,q}^s}$$

follows by analogous computations. Inserting these estimates into (2.3) the claim follows in case $\max(p, q) < \infty$. Remark that all computations can be done also by taking the l^∞ - and L^∞ -norm, respectively. The proof is complete. \square

Remark 2.16. (i) Some basic ideas of the above proof are taken over from Bourdaud [2], see also [4].

(ii) Also Wang, Lifeng, Boling [30] considered modulation spaces with an exponential weight. More exactly, they investigated the scale $E_{p,q}^\lambda(\mathbb{R}^n)$, defined as follows. A tempered distribution f belongs to $E_{p,q}^\lambda(\mathbb{R}^n)$ if

$$\|f\|_{E_{p,q}^\lambda} := \left(\sum_{k \in \mathbb{Z}^n} 2^{\lambda|k|q} \|\square_k f\|_{L^p}^q \right)^{1/q} < \infty.$$

For this scale they proved

$$\|f \cdot g\|_{E_{p,q}^\lambda} \leq c \|f\|_{E_{p_1, \min(1,q)}^\lambda} \|g\|_{E_{p_2, \min(1,q)}^\lambda}, \quad (2.6)$$

if $\lambda \geq 0$, $0 < p \leq p_1, p_2 \leq \infty$, $0 < q \leq \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let us mention that this is a result parallel to ours. In case $\lambda = (\log 2)^{-1}$ the spaces $E_{p,q}^\lambda(\mathbb{R}^n)$ coincide with $\mathcal{GM}_{p,q}^1(\mathbb{R}^n)$. In addition, if $\lambda > 0$ then we always have

$$E_{p,q}^\lambda(\mathbb{R}^n) \hookrightarrow \mathcal{GM}_{p,q}^s(\mathbb{R}^n), \quad s > 1.$$

This makes clear that (2.6) represents a borderline case with respect to Theorem 2.15.

Corollary 2.17. *Let $1 \leq p, q \leq \infty$ and $s > 1$. Then the Gevrey-modulation space $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ is an algebra under pointwise multiplication.*

Proof. As a consequence of Theorem 2.15 and Corollary 2.10 we obtain

$$\begin{aligned} \|f \cdot g\|_{\mathcal{GM}_{p,q}^s} &\leq C \|f\|_{\mathcal{GM}_{2p,q}^s} \|g\|_{\mathcal{GM}_{2p,q}^s} \\ &\leq C_1 \|f\|_{\mathcal{GM}_{p,q}^s} \|g\|_{\mathcal{GM}_{p,q}^s} \end{aligned} \quad (2.7)$$

with C_1 independent of f and g . Hence the claim follows. \square

Remark 2.18. (i) This time the constant C_1 depends on the window ρ , n , q , s and p .

(ii) Concerning the weighted modulation spaces $M_{p,q}^s(\mathbb{R}^n)$ there are several contributions to the algebra problem. We refer to Feichtinger [7], Iwabuchi [12] and Sugimoto et al [22].

(iii) In view of Lemma 2.13 Corollary 2.17 extends earlier results, obtained in [4] for $p = q = 2$, to the general case.

(iv) The restriction to values of $s > 1$ is not a technical one. In [4] the authors have shown that $\mathcal{GM}_{2,2}^1(\mathbb{R}^n)$ is not an algebra with respect to pointwise multiplication.

3. A Non-analytic Superposition Result on Gevrey-modulation Spaces

We need to proceed with some preparations. An essential tool in proving our main result will be a certain subalgebra property of the Gevrey-modulation spaces $\mathcal{GM}_{p,q}^s$. Therefore we consider the following decomposition of the phase space. Let $R > 0$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be fixed with $\epsilon_j \in \{0, 1\}$, $j = 1, \dots, n$. Then a decomposition of \mathbb{R}^n into $(2^n + 1)$ parts is given by

$$P_R := \{\xi \in \mathbb{R}^n : |\xi_j| \leq R, j = 1, \dots, n\}$$

and

$$P_R(\epsilon) := \{\xi \in \mathbb{R}^n : \text{sgn}(\xi_j) = (-1)^{\epsilon_j}, j = 1, \dots, n\} \setminus P_R.$$

For given p, q, s , $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $R > 0$ we introduce the spaces

$$\mathcal{GM}_{p,q}^s(\epsilon, R) := \{f \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n) : \text{supp } \mathcal{F}(f) \subset P_R(\epsilon)\}.$$

As above we will use the convention that for given q , $1 \leq q \leq \infty$, the number q' is defined by $\frac{1}{q} + \frac{1}{q'} = 1$.

Proposition 3.1. *Let $1 \leq p, q \leq \infty$, $s > 1$ and $R \geq 2$. We put $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$. Then, for any admissible ϵ*

$$\|f \cdot g\|_{\mathcal{GM}_{p,q}^s} \leq D_R \|f\|_{\mathcal{GM}_{p_1,q}^s} \|g\|_{\mathcal{GM}_{p_2,q}^s}$$

holds for all $f \in \mathcal{GM}_{p_1,q}^s(\epsilon, R)$ and all $g \in \mathcal{GM}_{p_2,q}^s(\epsilon, R)$, where the constant D_R is given by

$$D_R := C_0 \left(\int_{\delta q'(R-2)^{\frac{1}{s}}}^{\infty} y^{sn-1} e^{-y} dy \right)^{\frac{1}{q'}}.$$

Here $C_0 > 0$ denotes a constant which depends only on p, q, s and n .

Proof. Let $f \in \mathcal{GM}_{p_1,q}^s(\epsilon, R)$ and $g \in \mathcal{GM}_{p_2,q}^s(\epsilon, R)$. By

$$\text{supp}(\mathcal{F}f * \mathcal{F}g) \subset \{\xi + \eta : \xi \in \text{supp } \mathcal{F}(f), \eta \in \text{supp } \mathcal{F}(g)\}$$

we have $\text{supp } \mathcal{F}(fg) \subset P_R(\epsilon)$. In order to show the subalgebra property we follow the same steps as in the proof of Theorem 2.15. Let

$$P_R^*(\epsilon) := \left\{ k \in \mathbb{Z}^n : \|k\|_{\infty} > R - 1, \quad \text{sgn}(k_j) = (-1)^{\epsilon_j}, j = 1, \dots, n \right\}.$$

Hence, if $\text{supp } \sigma_k \cap P_R(\epsilon) \neq \emptyset$, then $k \in P_R^*(\epsilon)$ follows. In our situation the estimate (2.3) can be rewritten as

$$\begin{aligned} & \left(\sum_{k \in P_R^*(\epsilon)} e^{|k|^{\frac{1}{s}} q} \|\mathcal{F}^{-1}(\sigma_k \mathcal{F}(f \cdot g))\|_{L^p}^q \right)^{\frac{1}{q}} \\ & \leq c_2 \max_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \left(\sum_{k \in P_R^*(\epsilon)} e^{|k|^{\frac{1}{s}} q} \left[\sum_{\substack{l \in \mathbb{Z}^n, \\ l, t-(l-k) \in P_R^*(\epsilon)}} \|f_{t-(l-k)}\|_{L^{p_1}} \|g_l\|_{L^{p_2}} \right]^q \right)^{\frac{1}{q}}. \end{aligned}$$

According to this estimate we introduce the abbreviations

$$\begin{aligned} S_{1,t,k} &:= \sum_{\substack{l \in \mathbb{Z}^n: l, t-(l-k) \in P_R^*(\epsilon), \\ |l| \leq |l-k|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta |l|^{\frac{1}{s}}}, \\ S_{2,t,k} &:= \sum_{\substack{l \in \mathbb{Z}^n: l, t-(l-k) \in P_R^*(\epsilon), \\ |l-k| \leq |l|}} e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} e^{-\delta |l-k|^{\frac{1}{s}}} \end{aligned}$$

for all $k \in P_R^*(\epsilon)$. As above we conclude

$$\begin{aligned} S_{1,t,k} &\leq \left(\sum_{\substack{l \in \mathbb{Z}^n, \\ l, t-(l-k) \in P_R^*(\epsilon), \\ |l| \leq |l-k|}} \left| e^{|l-k|^{\frac{1}{s}}} \|f_{t-(l-k)}\|_{L^{p_1}} e^{|l|^{\frac{1}{s}}} \|g_l\|_{L^{p_2}} \right|^q \right)^{\frac{1}{q}} \\ &\quad \times \left(\sum_{\substack{l \in \mathbb{Z}^n: l, t-(l-k) \in P_R^*(\epsilon), \\ |l| \leq |l-k|}} e^{-\delta q' |l|^{\frac{1}{s}}} \right)^{\frac{1}{q'}}. \end{aligned}$$

By definition of the set $P_R^*(0, \dots, 0)$ we find in case $R \geq 2$

$$\begin{aligned} \sum_{\substack{l \in \mathbb{Z}^n, \\ l, t-(l-k) \in P_R^*(0, \dots, 0), \\ |l| \leq |l-k|}} e^{-\delta q' |l|^{\frac{1}{s}}} &\leq \sum_{l \in \mathbb{Z}^n: \|l\|_{\infty} > R-1} e^{-\delta q' |l|^{\frac{1}{s}}} \\ &\leq \sum_{\substack{l \in \mathbb{Z}^n, \\ \|l\|_{\infty} > R-1}} \int_{\substack{x \in \mathbb{R}^n, \\ l_i-1 \leq x_i \leq l_i, \\ i=1, \dots, n}} e^{-\delta q' |x|^{\frac{1}{s}}} dx \\ &\leq \int_{\|x\|_{\infty} > R-2} e^{-\delta q' |x|^{\frac{1}{s}}} dx \\ &\leq \int_{|x| > R-2} e^{-\delta q' |x|^{\frac{1}{s}}} dx. \end{aligned}$$

A symmetry argument yields the same estimate in case $\epsilon \neq (0, \dots, 0)$. By means of some simple calculations we obtain

$$\begin{aligned}
 \int_{|x|>R-2} e^{-\delta q'|x|^{\frac{1}{s}}} dx &= 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \int_{R-2}^{\infty} r^{n-1} e^{-\delta q' r^{\frac{1}{s}}} dr \\
 &\stackrel{[t=r^{\frac{1}{s}}]}{=} 2 \frac{\pi^{n/2}}{\Gamma(n/2)} s \int_{(R-2)^{\frac{1}{s}}}^{\infty} t^{sn-1} e^{-\delta q' t} dt \\
 &\stackrel{[y=\delta q' t]}{=} 2 \frac{\pi^{n/2}}{\Gamma(n/2)} s (\delta q')^{-sn} \int_{\delta q' (R-2)^{\frac{1}{s}}}^{\infty} y^{sn-1} e^{-y} dy \\
 &=: E_R.
 \end{aligned} \tag{3.1}$$

Hence

$$\left(\sum_{k \in P_R^*(\epsilon)} S_{1,t,k}^q \right)^{1/q} \leq E_R^{1/q'} \left(\sum_{j \in \mathbb{Z}^n} e^{|j|^{\frac{1}{s}} q} \|f_{t-j}\|_{L^{p_1}}^q \sum_{k \in \mathbb{Z}^n} e^{|j+k|^{\frac{1}{s}} q} \|g_{j+k}\|_{L^{p_2}}^q \right)^{\frac{1}{q}}.$$

With c_4 defined as above this implies

$$\left(\sum_{k \in \mathbb{Z}^n} S_{1,t,k}^q \right)^{1/q} \leq E_R^{1/q'} c_4 \|g\|_{\mathcal{GM}_{p_2,q}^s} \|f\|_{\mathcal{GM}_{p_1,q}^s}.$$

For the second sum the estimate

$$\left(\sum_{k \in \mathbb{Z}^n} S_{2,t,k}^q \right)^{1/q} \leq E_R^{1/q'} c_5 \|g\|_{\mathcal{GM}_{p_2,q}^s} \|f\|_{\mathcal{GM}_{p_1,q}^s}$$

follows by analogous computations. \square

Arguing as in proof of Corollary 2.17 we obtain the following.

Corollary 3.2. *Let $1 \leq p, q \leq \infty$ and $s > 1$. For $R \geq 2$ and any ϵ the spaces $\mathcal{GM}_{p,q}^s(\epsilon, R)$ are subalgebras of $\mathcal{GM}_{p,q}^s$. Furthermore, it holds*

$$\|f \cdot g\|_{\mathcal{GM}_{p,q}^s} \leq F_R \|f\|_{\mathcal{GM}_{p,q}^s} \|g\|_{\mathcal{GM}_{p,q}^s}$$

for all $f, g \in \mathcal{GM}_{p,q}^s(\epsilon, R)$. The constant F_R can be specified as

$$F_R := C_1 \left(\int_{\delta q' (R-2)^{\frac{1}{s}}}^{\infty} y^{sn-1} e^{-y} dy \right)^{\frac{1}{q'}},$$

where the constant $C_1 > 0$ depends only on p, q, s and n .

Note that in the following we assume every function to be real-valued unless it is explicitly stated that complex-valued functions are allowed.

In order to establish the next result we need to recall some lemmas. The first one concerns a standard estimate of Fourier multipliers, see, e.g., [25, Theorem 1.5.2]. By $H^s(\mathbb{R}^n)$ we denote the Sobolev space of fractional order s built on $L_2(\mathbb{R}^n)$.

Lemma 3.3. *Let $1 \leq r \leq \infty$ and assume that $s > \frac{n}{2}$. Then there exists a constant $C > 0$ such that*

$$\|\mathcal{F}^{-1}(\phi \mathcal{F}f)\|_{L^r} \leq C \|\phi\|_{H^s} \|f\|_{L^r}$$

holds for all $f \in L^r(\mathbb{R}^n)$ and all $\phi \in H^s(\mathbb{R}^n)$.

The next two technical lemmas have been taken from [4].

Lemma 3.4. *Let $N \in \mathbb{N}$ and suppose a_1, a_2, \dots, a_N to be complex numbers. Then it holds*

$$a_1 \cdot a_2 \cdot \dots \cdot a_N - 1 = \sum_{l=1}^N \sum_{\substack{j=(j_1, \dots, j_l), \\ 0 \leq j_1 < \dots < j_l \leq N}} (a_{j_1} - 1) \cdot \dots \cdot (a_{j_l} - 1).$$

Proof. Cf. Lemma 4.6. in [4]. □

Lemma 3.5. *Let $\alpha > 0$. Define*

$$f(t) := \int_t^\infty e^{-y} y^{\alpha-1} dy, \quad t \geq 0.$$

The inverse g of the function f maps $(0, \Gamma(\alpha)]$ onto $[0, \infty)$ and it holds

$$\lim_{u \downarrow 0} \frac{g(u)}{\log \frac{1}{u}} = 1.$$

Proof. Cf. Lemma 4.5. in [4]. □

The non-analytic superposition, which will be stated in Theorem 3.9, is based on the following lemma.

Lemma 3.6. *Let $s > 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$. Then it holds*

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq c \|u\|_{\mathcal{GM}_{p,q}^s} \begin{cases} e^{b\|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}} \log \|u\|_{\mathcal{GM}_{p,q}^s}} & \text{if } \|u\|_{\mathcal{GM}_{p,q}^s} > 1, \\ 1 & \text{if } \|u\|_{\mathcal{GM}_{p,q}^s} \leq 1 \end{cases}$$

with constants $b, c > 0$ independent of u .

Proof. This proof basically follows the same steps as the proof of Theorem 2.3 in [4].

Step 1. Let $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ satisfying $\text{supp } \mathcal{F}(u) \subset P_R$ for some $R \geq 2$. First we consider the Taylor expansion

$$e^{iu} - 1 = \sum_{l=1}^r \frac{(iu)^l}{l!} + \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!}$$

resulting in the norm estimate

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq \left\| \sum_{l=1}^r \frac{(iu)^l}{l!} \right\|_{\mathcal{GM}_{p,q}^s} + \left\| \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!} \right\|_{\mathcal{GM}_{p,q}^s} =: S_1 + S_2.$$

By Corollary 2.17, see in particular (2.7), we obtain

$$S_2 \leq \sum_{l=r+1}^{\infty} \frac{1}{l!} \|u^l\|_{\mathcal{GM}_{p,q}^s} \leq \frac{1}{C_1} \sum_{l=r+1}^{\infty} \frac{(C_1 \|u\|_{\mathcal{GM}_{p,q}^s})^l}{l!}.$$

Now we choose r as a function of $\|u\|_{\mathcal{GM}_{p,q}^s}$ and distinguish two cases:

1. $C_1 \|u\|_{\mathcal{GM}_{p,q}^s} > 1$. Assume that

$$3 C_1 \|u\|_{\mathcal{GM}_{p,q}^s} \leq r \leq 3 C_1 \|u\|_{\mathcal{GM}_{p,q}^s} + 1 \quad (3.2)$$

and recall Stirling's formula $l! = \Gamma(l+1) \geq l^l e^{-l} \sqrt{2\pi l}$. Thus, we get

$$\begin{aligned} \sum_{l=r+1}^{\infty} \frac{(C_1 \|u\|_{\mathcal{GM}_{p,q}^s})^l}{l!} &\leq \sum_{l=r+1}^{\infty} \left(\frac{r}{l}\right)^l \left(\frac{e}{3}\right)^l \frac{1}{\sqrt{2\pi l}} \\ &\leq \sum_{l=r+1}^{\infty} \left(\frac{e}{3}\right)^l \leq \frac{3}{3-e}. \end{aligned}$$

2. $C_1 \|u\|_{\mathcal{GM}_{p,q}^s} \leq 1$. It follows

$$\sum_{l=r+1}^{\infty} \frac{(C_1 \|u\|_{\mathcal{GM}_{p,q}^s})^l}{l!} \leq \sum_{l=1}^{\infty} \frac{(C_1 \|u\|_{\mathcal{GM}_{p,q}^s})^l}{l!} \leq C_1 e \|u\|_{\mathcal{GM}_{p,q}^s}.$$

Both together can be summarized as

$$S_2 \leq C_2 \|u\|_{\mathcal{GM}_{p,q}^s}, \quad C_2 := \max\left(C_1 e, \frac{3}{3-e}\right). \quad (3.3)$$

To estimate S_1 we check the support of $\mathcal{F}u^\ell$ and find

$$\begin{aligned} S_1 &= \left\| \sum_{l=1}^r \frac{(1u)^l}{l!} \right\|_{\mathcal{GM}_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} e^{|k|^{\frac{1}{s}} q} \left\| \square_k \left(\sum_{l=1}^r \frac{(1u)^l}{l!} \right) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{|k|^{\frac{1}{s}} q} \left\| \square_k \left(\sum_{l=1}^r \frac{(1u)^l}{l!} \right) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\ &\leq \left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{|k|^{\frac{1}{s}} q} \left\| \square_k (e^{1u} - 1) \right\|_{L^p}^q \right)^{\frac{1}{q}} + S_2. \end{aligned}$$

Concerning S_2 we proceed as above. To estimate the first part we observe that

$$C_3 := \sup_{k \in \mathbb{Z}^n} \|\sigma_k\|_{H^t} = \|\sigma_0\|_{H^t} < \infty,$$

see Lemma 3.3. Furthermore, \cos, \sin are Lipschitz continuous and consequently we get

$$\begin{aligned} \|\square_k(e^{1u} - 1)\|_{L^p} &\leq C_3 \|e^{1u} - 1\|_{L^p} \\ &\leq C_3 (\|\cos u - \cos 0\|_{L^p} + \|\sin u - \sin 0\|_{L^p}) \\ &\leq 2 C_3 \|u - 0\|_{L^p}. \end{aligned}$$

This implies

$$\begin{aligned} & \left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{|k|^{\frac{1}{s}} q} \|\square_k(e^{iu} - 1)\|_{L^p}^q \right)^{\frac{1}{q}} \\ & \leq 2 C_3 \|u\|_{L^p} \left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{|k|^{\frac{1}{s}} q} \right)^{\frac{1}{q}}. \end{aligned}$$

With a calculation similar to (3.1) we derive

$$\begin{aligned} \sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{|k|^{\frac{1}{s}} q} & \leq \int_{\|x\|_\infty < Rr+1} e^{|x|^{\frac{1}{s}} q} dx \\ & \leq \int_{|x| < \sqrt{n}(Rr+1)} e^{|x|^{\frac{1}{s}} q} dx \\ & = 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^{\sqrt{n}(Rr+1)} \tau^{n-1} e^{\tau^{\frac{1}{s}} q} d\tau \\ & \leq 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{s}{q} (\sqrt{n}(Rr+1))^{n-\frac{1}{s}} e^{(\sqrt{n}(Rr+1))^{\frac{1}{s}} q}. \quad (3.4) \end{aligned}$$

To simplify notation we define

$$C_4 := \sup_{r \in \mathbb{N}} \sup_{R \geq 2} \left(2 \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{s}{q} n^{(n-\frac{1}{s})/2} e^{-(\sqrt{n}(Rr+1))^{\frac{1}{s}} q} \right)^{1/q}.$$

In addition we shall use

$$\|u\|_{L^p} \leq C_5 \|u\|_{\mathcal{GM}_{p,q}^s}, \quad C_5 := \left(\sum_{k \in \mathbb{Z}^n} e^{-|k|^{1/s} q'} \right)^{1/q'}$$

which follows from Hölder's inequality. Summarizing we have found

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq \left(2 C_2 + 2 C_5 C_4 C_3 e^{2(\sqrt{n}(Rr+1))^{\frac{1}{s}} q} \right) \|u\|_{\mathcal{GM}_{p,q}^s},$$

where the chosen constants depend on the dimension n , the weight parameter s and the integrability parameters p and q . Next we apply (3.2) which results in

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq c_0 \|u\|_{\mathcal{GM}_{p,q}^s} \left(1 + e^{b_0 R^{\frac{1}{s}} \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}}} \right), \quad (3.5)$$

valid for all $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ satisfying $\text{supp } \mathcal{F}(u) \subset P_R$ and with positive constants b_0, c_0 depending on n, s and q but independent of u, r and R .

Step 2. The next step consists of choosing general $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$. Here we need the restriction $1 < p < \infty$. For those p the characteristic functions χ of cubes are Fourier multipliers in $L^p(\mathbb{R}^n)$ by the famous Riesz Theorem and therefore also in $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$. In addition we shall make use of the fact that the norm of the operator $f \mapsto \mathcal{F}^{-1} \chi \mathcal{F} f$ does not depend on the size of the cube.

Below we shall denote this norm by $C_6 = C_6(p)$. We refer to Lizorkin [14] for all details. For decomposing u on the phase space we introduce functions $\chi_{R,\epsilon}$ and χ_R , that is, the characteristic functions of the sets $P_R(\epsilon)$ and P_R , respectively. By defining

$$\begin{aligned} u_\epsilon(x) &= \mathcal{F}^{-1}[\chi_{R,\epsilon}(\xi)(\mathcal{F}u)(\xi)](x), \\ u_0(x) &= \mathcal{F}^{-1}[\chi_R(\xi)(\mathcal{F}u)(\xi)](x) \end{aligned}$$

we can rewrite u as

$$u(x) = u_0(x) + \sum_{\epsilon \in I} u_\epsilon(x), \quad (3.6)$$

where I is the set of all $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_j \in \{0, 1\}$, $j = 1, \dots, n$. Hence

$$\|u\|_{\mathcal{GM}_{p,q}^s} \leq \|u_0\|_{\mathcal{GM}_{p,q}^s} + \sum_{\epsilon \in I} \|u_\epsilon\|_{\mathcal{GM}_{p,q}^s} \quad (3.7)$$

and

$$\max \left(\|u_0\|_{\mathcal{GM}_{p,q}^s}, \|u_\epsilon\|_{\mathcal{GM}_{p,q}^s} \right) \leq C_6 \|u\|_{\mathcal{GM}_{p,q}^s}.$$

Due to the representation (3.6) and using an appropriate enumeration Lemma 3.4 leads to

$$e^{iu} - 1 = \sum_{l=1}^{2^n+1} \sum_{0 \leq j_1 < \dots < j_l \leq 2^n} (e^{iu_{j_1}} - 1) \cdot \dots \cdot (e^{iu_{j_l}} - 1).$$

Corollary 2.17 immediately yields

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq \sum_{l=1}^{2^n+1} C_1^{l-1} \sum_{0 \leq j_1 < \dots < j_l \leq 2^n} \|e^{iu_{j_1}} - 1\|_{\mathcal{GM}_{p,q}^s} \cdot \dots \cdot \|e^{iu_{j_l}} - 1\|_{\mathcal{GM}_{p,q}^s}. \quad (3.8)$$

By Corollary 3.2, (3.7) and (3.5) it follows

$$\begin{aligned} \|e^{iu_{j_k}} - 1\|_{\mathcal{GM}_{p,q}^s} &= \left\| \sum_{l=1}^{\infty} \frac{(iu_{j_k})^l}{l!} \right\|_{\mathcal{GM}_{p,q}^s} \leq \frac{1}{F_R} \left(e^{F_R \|u_{j_k}\|_{\mathcal{GM}_{p,q}^s}} - 1 \right) \\ &\leq \frac{1}{F_R} \left(e^{F_R C_6 \|u\|_{\mathcal{GM}_{p,q}^s}} - 1 \right), \end{aligned} \quad (3.9)$$

as well as

$$\|e^{iu_0} - 1\|_{\mathcal{GM}_{p,q}^s} \leq c_0 C_6 \|u\|_{\mathcal{GM}_{p,q}^s} \left(1 + e^{b_0 C_6^{\frac{1}{s}} R^{\frac{1}{s}} \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}}} \right), \quad (3.10)$$

where we used the Fourier multiplier assertion mentioned at the beginning of this substep. The final step in our proof is to choose the number R as a function of $\|u\|_{\mathcal{GM}_{p,q}^s}$ such that (3.9) and (3.10) will be approximately of the same size.

Substep 2.1. Let $\|u\|_{\mathcal{GM}_{p,q}^s} \leq 1$. We choose $R = 3$. Then (3.8) combined with (3.9) and (3.10) results in the estimate

$$\|e^{iu} - 1\|_{\mathcal{GM}_{p,q}^s} \leq C_7 \|u\|_{\mathcal{GM}_{p,q}^s}, \quad (3.11)$$

where C_7 does not depend on u .

Substep 2.2. Let $\|u\|_{\mathcal{GM}_{p,q}^s} > 1$. As mentioned in Corollary 3.2 we know that the algebra constant F_R in (3.9) is a function of R , i.e.,

$$F_R := C_1 \left(\int_{\delta q'(R-2)^{\frac{1}{s}}}^{\infty} y^{sn-1} e^{-y} dy \right)^{\frac{1}{q'}}.$$

Taking into account that

- F_R (as a function of R) is strictly decreasing and positive and
- $\lim_{R \rightarrow \infty} F_R = 0$

we can easily set

$$\frac{F_R}{F_2} = \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}-1}$$

for some $R > 2$. In view of Lemma 3.5 this gives

$$C_8 \left(f(\delta q'(R-2)^{\frac{1}{s}}) \right)^{\frac{1}{q'}} = \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}-1}$$

with an appropriate positive constant C_8 . Thus, by Lemma 3.5 it follows

$$R = 2 + (\delta q')^{-s} \left(g \left(\frac{\|u\|_{\mathcal{GM}_{p,q}^s}^{q'(\frac{1}{s}-1)}}{C_8^{q'}} \right) \right)^s$$

and, moreover,

$$R \leq C_9 + C_{10} \left(q' - \frac{q'}{s} \right)^s \log^s \|u\|_{\mathcal{GM}_{p,q}^s}.$$

Note that the constants C_9 and C_{10} are independent of u . Now (3.8) combined with (3.9) and (3.10) results in

$$\begin{aligned} & \|e^{1u} - 1\|_{\mathcal{GM}_{p,q}^s} \\ & \leq C_{11} \max_{\alpha \in \{0,1\}, \beta \in \{0, \dots, 2^n\}} \left(c_0 C_6 \|u\|_{\mathcal{GM}_{p,q}^s} \left(1 + e^{b_0 C_6^{\frac{1}{s}} R^{\frac{1}{s}} \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}}} \right) \right)^\alpha \\ & \quad \times \left(\frac{e^{F_R C_6 \|u\|_{\mathcal{GM}_{p,q}^s} - 1}}{F_R} \right)^\beta \\ & \leq C_{12} \max_{\alpha \in \{0,1\}, \beta \in \{0, \dots, 2^n\}} \left(\|u\|_{\mathcal{GM}_{p,q}^s} \left(1 + e^{b_1 \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}} \log \|u\|_{\mathcal{GM}_{p,q}^s}} \right) \right)^\alpha \\ & \quad \times \|u\|_{\mathcal{GM}_{p,q}^s}^{\beta(1-\frac{1}{s})} \left(\frac{e^{F_2 C_6 \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}} - 1}}{F_2} \right)^\beta \\ & \leq C_{13} \|u\|_{\mathcal{GM}_{p,q}^s} \left(1 + e^{b \|u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}} \log \|u\|_{\mathcal{GM}_{p,q}^s}} \right) \end{aligned} \tag{3.12}$$

with a constant C_{13} independent of u . \square

Remark 3.7. The restriction of p to the interval $(1, \infty)$ is caused by our decomposition technique, see Step 2 of the preceeding proof. We do not know whether Lemma 3.6 extends to $p = 1$ and/or $p = \infty$.

Lemma 3.8. *Let $s > 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$.*

- (i) *The mapping $u \mapsto e^{iu} - 1$ is locally Lipschitz continuous (considered as a mapping of $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ into $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$).*
- (ii) *Assume $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ to be fixed and define a function $g : \mathbb{R} \mapsto \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ by $g(\xi) = e^{iu(x)\xi} - 1$. Then the function g is continuous.*

Proof. Local Lipschitz continuity follows from the identity

$$e^{iu} - e^{iv} = (e^{iv} - 1)(e^{i(u-v)} - 1) + (e^{i(u-v)} - 1), \quad (3.13)$$

the algebra property of $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ and Lemma 3.6.

To prove the continuity of g we also employ the identity (3.13). The claim follows by using the algebra property and Lemma 3.6. \square

Now we can establish the announced superposition result.

Theorem 3.9. *Let $s > 1$, $1 < p < \infty$, $1 \leq q \leq \infty$ and μ be a complex measure on \mathbb{R} such that*

$$L_1(\lambda) = \int_{\mathbb{R}} e^{\lambda(|\xi|^{\frac{1}{s}} \log |\xi|)} d|\mu|(\xi) < \infty \quad (3.14)$$

for any $\lambda > 0$ and such that $\mu(\mathbb{R}) = 0$.

Furthermore, assume that the function f is the inverse Fourier transform of μ . Then $f \in C^\infty$ and the composition operator $T_f : u \mapsto f \circ u$ maps $\mathcal{GM}_{p,q}^s$ into $\mathcal{GM}_{p,q}^s$.

Proof. Equation (3.14) yields $\int_{\mathbb{R}} d|\mu|(\xi) < \infty$. Thus, μ is a finite measure and $\mu(\mathbb{R}) = 0$ makes sense. Now we define the inverse Fourier transform of μ

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi t} d\mu(\xi).$$

Moreover, $\int_{\mathbb{R}} |(i\xi)^j| d|\mu|(\xi) < \infty$ is deduced from equation (3.14) for all $j \in \mathbb{N}$. This gives $f \in C^\infty$ and due to $\mu(\mathbb{R}) = 0$ we can also write f as follows:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (e^{i\xi t} - 1) d\mu(\xi).$$

Since μ is a complex measure we can split it up into real part μ_r and imaginary part μ_i , where each of them is a signed measure. Without loss of generality we proceed our computations only with the positive real measure μ_r^+ . For all measurable sets E we have $\mu_r^+(E) \leq |\mu|(E)$.

Let $u \in \mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ and define the function $g(\xi) = e^{iu(x)\xi} - 1$ analogously to Lemma 3.8. Then g is Bochner integrable because of its continuity and taking into account that the measure μ_r^+ is finite. Therefore we obtain the Bochner integral

$$\int_{\mathbb{R}} (e^{iu(x)\xi} - 1) d\mu_r^+(\xi) = \int_{\mathbb{R}} g(\xi) d\mu_r^+(\xi)$$

with values in $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$. By applying Minkowski inequality it follows

$$\left\| \int_{\mathbb{R}} (e^{iu(\cdot)\xi} - 1) d\mu_r^+(\xi) \right\|_{\mathcal{GM}_{p,q}^s} \leq \int_{\mathbb{R}} \|e^{iu(\cdot)\xi} - 1\|_{\mathcal{GM}_{p,q}^s} d|\mu|(\xi).$$

Using the abbreviation $\|u\| := \|u\|_{\mathcal{GM}_{p,q}^s}$, Lemma 3.6 together with equation (3.14) yields

$$\begin{aligned} \int_{|\xi| \|u\| \geq 1} \|e^{iu(\cdot)\xi} - 1\|_{\mathcal{GM}_{p,q}^s} d|\mu|(\xi) &\leq c \int_{|\xi| \|u\| \geq 1} e^{b\|\xi u\|_{\mathcal{GM}_{p,q}^s}^{\frac{1}{s}} \log \|\xi u\|_{\mathcal{GM}_{p,q}^s}} d|\mu|(\xi) \\ &< \infty. \end{aligned}$$

In this way also the remaining part of the integral $|\xi| \leq 1/\|u\|$ can be treated. The same estimates also hold for the measures μ_r^- , μ_i^+ and μ_i^- . Thus, the result is obtained by

$$\begin{aligned} \|\sqrt{2\pi}f(u(x))\|_{\mathcal{GM}_{p,q}^s} &= \left\| \int_{\mathbb{R}} g(\xi) d\mu_r^+ - \int_{\mathbb{R}} g(\xi) d\mu_r^- \right. \\ &\quad \left. + \int_{\mathbb{R}} g(\xi) d\mu_i^+ - \int_{\mathbb{R}} g(\xi) d\mu_i^- \right\|_{\mathcal{GM}_{p,q}^s} \\ &\leq \int_{\mathbb{R}} \|g(\xi)\|_{\mathcal{GM}_{p,q}^s} d|\mu_r^+| + \int_{\mathbb{R}} \|g(\xi)\|_{\mathcal{GM}_{p,q}^s} d|\mu_r^-| \\ &\quad + \int_{\mathbb{R}} \|g(\xi)\|_{\mathcal{GM}_{p,q}^s} d|\mu_i^+| + \int_{\mathbb{R}} \|g(\xi)\|_{\mathcal{GM}_{p,q}^s} d|\mu_i^-|, \end{aligned}$$

where every integral on the right-hand side is finite. Thus, the statement is proved. \square

Remark 3.10. Theorem 3.9 is an extension of earlier results obtained for $\mathcal{GM}_{2,2}^s(\mathbb{R}^n)$ in [4].

For practical reasons we remark the following consequence.

Corollary 3.11. *Let the weight parameter $s > 1$, $1 < p < \infty$, $1 \leq q \leq \infty$ and μ be a complex measure on \mathbb{R} with the corresponding bounded density function g , i.e., $d\mu(\xi) = g(\xi) d\xi$. Suppose that*

$$\lim_{|\xi| \rightarrow \infty} \frac{|\xi|^{\frac{1}{s}} \log |\xi|}{\log |g(\xi)|} = 0 \quad (3.15)$$

and $\int_{\mathbb{R}} d\mu(\xi) = \int_{\mathbb{R}} g(\xi) d\xi = 0$. Assume the function f to be the inverse Fourier transform of g . Then $f \in C^\infty$ and the composition operator $T_f : u \mapsto f \circ u$ maps $\mathcal{GM}_{p,q}^s$ into $\mathcal{GM}_{p,q}^s$.

Proof. Most of the work has been done in the proof of Theorem 3.9. The condition (3.15) yields that the modulus of $\lim_{|\xi| \rightarrow \infty} \log |g(\xi)|$ needs to be infinity. This fact together with the boundedness of g implies $\lim_{|\xi| \rightarrow \infty} g(\xi) = 0$. Moreover, for any $\lambda > 0$ there exists a number $N > 0$ such that

$$-\frac{|\xi|^{\frac{1}{s}} \log |\xi|}{\log |g(\xi)|} \leq \frac{1}{2\lambda}, \quad |\xi| > N.$$

Thus, we obtain

$$|g(\xi)| \leq e^{-2\lambda |\xi|^{\frac{1}{s}} \log |\xi|}, \quad |\xi| > N,$$

and it follows

$$\begin{aligned} \int_{|\xi|>N} e^{\lambda(|\xi|^{\frac{1}{s}} \log |\xi|)} d|\mu|(\xi) &= \int_{|\xi|>N} e^{\lambda(|\xi|^{\frac{1}{s}} \log |\xi|)} |g(\xi)| d\xi \\ &\leq \int_{|\xi|>N} e^{-\lambda(|\xi|^{\frac{1}{s}} \log |\xi|)} d\xi < \infty. \end{aligned}$$

This completes the proof. \square

One Example

We recall a construction considered in Rodino [17, Example 1.4.9]. Let $\mu < 0$ and let for $t \in \mathbb{R}$

$$\psi_\mu(t) := \begin{cases} e^{-t^\mu} & \text{if } t > 0; \\ 0 & \text{otherwise.} \end{cases}$$

By taking

$$\phi_\mu(t) := \psi_\mu(1-t) \cdot \psi_\mu(t), \quad t \in \mathbb{R},$$

we obtain a compactly supported C^∞ function on \mathbb{R} . It follows $\varphi_\mu \in G^s(\mathbb{R})$ for $s = 1 - 1/\mu$. Here the classes $G^s(\mathbb{R})$ refer to classical Gevrey regularity, see [17, Definition 1.4.1]. We skip the definition here and recall a further result, see [17, Thm. 1.6.1]. Since $\varphi_\mu \in G^s(\mathbb{R})$ has compact support there exists a positive constant c and some $\varepsilon > 0$ such that

$$|\mathcal{F}\varphi_\mu(\xi)| \leq c e^{-\varepsilon |\xi|^{1/s}}, \quad \xi \in \mathbb{R}.$$

Because of $\phi_\mu(0) = 0$ Cor. 3.11 yields the following.

Corollary 3.12. *Let the weight parameter $s > 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose that $\mu < \frac{1}{1-s}$. Then $T_{\varphi_\mu} : u \mapsto \varphi_\mu \circ u$ maps $\mathcal{GM}_{p,q}^s$ into $\mathcal{GM}_{p,q}^s$.*

4. A Non-analytic Superposition Result on Special Modulation Spaces

In the previous sections the results in [4] gave the motivation to use weights of Gevrey type. Now we want to leave the Gevrey frame and approach weights of Sobolev type. For brevity we put

$$\langle x \rangle_* := (e^{2e} + |x|^2)^{1/2}, \quad x \in \mathbb{R}^n.$$

Let

$$w(x) := e^{\log \langle x \rangle_* \log \log \langle x \rangle_*}, \quad x \in \mathbb{R}^n.$$

By means of our normalization we have $\log \log \langle x \rangle_* \geq 1$. Clearly, in case $t > 0$ and $s > 1$, there exist positive constants A_t and B_s such that

$$A_t \langle x \rangle_*^t \leq w(x) \leq B_s e^{|x|^{1/s}}, \quad x \in \mathbb{R}^n.$$

The spaces defined by such a weight may serve as a prototype for weighted modulation spaces where the weight is increasing stronger than any polynomial but weaker than $e^{|x|^{1/s}}$.

Definition 4.1. The integrability parameters are given by $1 \leq p, q \leq \infty$. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a fixed window. Then the modulation space $\mathcal{UM}^{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{UM}^{p,q}} := \left(\sum_{k \in \mathbb{Z}^n} e^{q \log \langle k \rangle_* \log \log \langle k \rangle_*} \|\mathcal{F}^{-1}(\square_k \mathcal{F} f)\|_{L^p}^q \right)^{\frac{1}{q}} < \infty$$

with obvious modifications when $p = \infty$ and/or $q = \infty$.

Remark 4.2. In fact, the modulation spaces $\mathcal{UM}^{p,q}(\mathbb{R}^n)$ contain all Gevrey-modulation spaces $\mathcal{GM}_{p,q}^s(\mathbb{R}^n)$ but are contained in every classical modulation space $M_{p,q}^s(\mathbb{R}^n)$, i.e.

$$\bigcup_{s>1} \mathcal{GM}_{p,q}^s(\mathbb{R}^n) \subset \mathcal{UM}^{p,q}(\mathbb{R}^n) \subset \bigcap_{s \in \mathbb{R}} M_{p,q}^s(\mathbb{R}^n).$$

By the same arguments as in Lemma 2.8 one can prove the following statements.

Lemma 4.3. (i) *The modulation space $\mathcal{UM}^{p,q}(\mathbb{R}^n)$ is a Banach space.*
(ii) *$\mathcal{UM}^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window $\rho \in C_0^\infty(\mathbb{R}^n)$ in the sense of equivalent norms.*
(iii) *$\mathcal{UM}^{p,q}(\mathbb{R}^n)$ has the Fatou property, i.e., if $(f_m)_{m=1}^\infty \subset \mathcal{UM}^{p,q}(\mathbb{R}^n)$ is a sequence such that $f_m \rightharpoonup f$ (weak convergence in $S'(\mathbb{R}^n)$) and*

$$\sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{UM}^{p,q}} < \infty,$$

then $f \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$ follows and

$$\|f\|_{\mathcal{UM}^{p,q}} \leq \sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{UM}^{p,q}} < \infty.$$

In order to prove that $\mathcal{UM}^{p,q}$ is an algebra under pointwise multiplication we need a counterpart of Lemma 2.14. Therefore we start with some elementary analysis. Let

$$w_*(t) := \log \langle t \rangle_* \log \log \langle t \rangle_*, \quad t \in [0, \infty).$$

This function is strongly increasing and its range is given by $[e, \infty)$. To proceed as in Section 3 we need to prove a counterpart of Lemma 2.14.

Lemma 4.4. *There exists a positive real number $s \in (0, 1)$ such that*

$$w_*(x) \leq w_*(y) + w_*(x - y) - s \min(w_*(y), w_*(x - y)) \quad (4.1)$$

holds for all $x, y \geq 0$.

Proof. Step 1. Preliminaries. We need some auxiliary functions and their basic properties. Obviously we have

$$w'_*(t) = \frac{t}{\langle t \rangle_*^2} (1 + \log \log \langle t \rangle_*), \quad t > 0,$$

and

$$w''_*(t) = \frac{1}{\langle t \rangle_*^2} (1 + \log \log \langle t \rangle_*) + \frac{t^2}{\langle t \rangle_*^4} \left(\frac{1}{\log \langle t \rangle_*} - 2 - 2 \log \log \langle t \rangle_* \right), \quad t > 0.$$

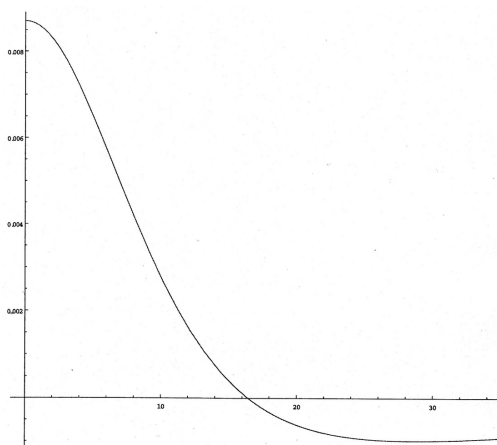


Figure 1

The plot of w''_* , see Figure 1, makes clear that there exists a positive number t_0 such that

$$\begin{aligned} w''_*(t_0) &= 0, \\ w''_*(t) &< 0 \quad \text{for } t > t_0, \\ w''_*(t) &> 0 \quad \text{for } 0 < t < t_0. \end{aligned}$$

A closer look shows that $t_0 \in (16.4449, 16.4451)$. Thus, w'_* has its global maximum at $t = t_0$.

Later on we shall need the following function

$$\begin{aligned} p(t) &:= \frac{t}{w_*(t)} w'_*(t) \\ &= \frac{t}{\log\langle t \rangle_* \log \log\langle t \rangle_*} \frac{t}{\langle t \rangle_*^2} (1 + \log \log\langle t \rangle_*) \\ &= \frac{t^2}{\langle t \rangle_*^2} \frac{1 + \log \log\langle t \rangle_*}{\log\langle t \rangle_* \log \log\langle t \rangle_*}. \end{aligned}$$

Clearly, $p_0 := \sup_{t>0} p(t) < 1$. Figure 2 shows that $p_0 \in (0, 410247, 0, 410248)$. In addition we need

$$q(t) := \frac{t}{w_*(t)}, \quad t > 0.$$

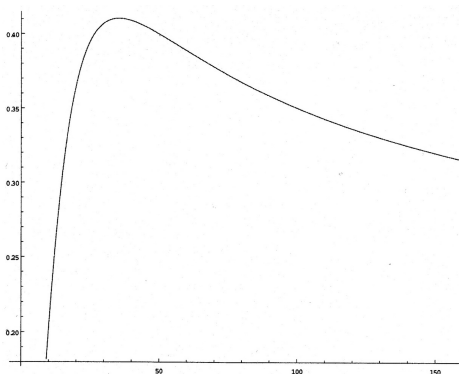


Figure 2

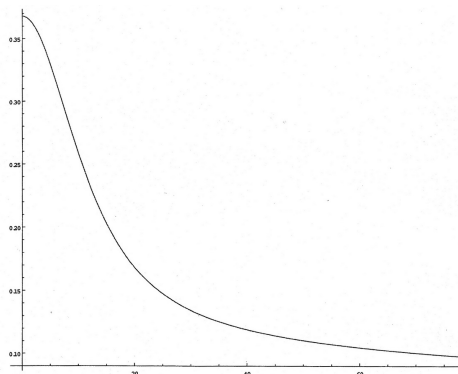


Figure 3

Considering the derivative q' , see Figure 3, it becomes clear that q is strictly increasing for all $t \geq 0$.

Step 2. We shall prove (4.1) in case $y \geq x$. From w_* increasing and $\min(w_*(y), w_*(x-y)) = w_*(x-y)$ we derive the validity of (4.1) for all s , $0 < s \leq 1$.

Step 3. Now we turn to the case $x > y$. We shall split our investigations into the three cases $x \geq 2t_0$, $t_0 < x < 2t_0$ and $0 \leq x \leq t_0$, where t_0 is the root of w''_* .

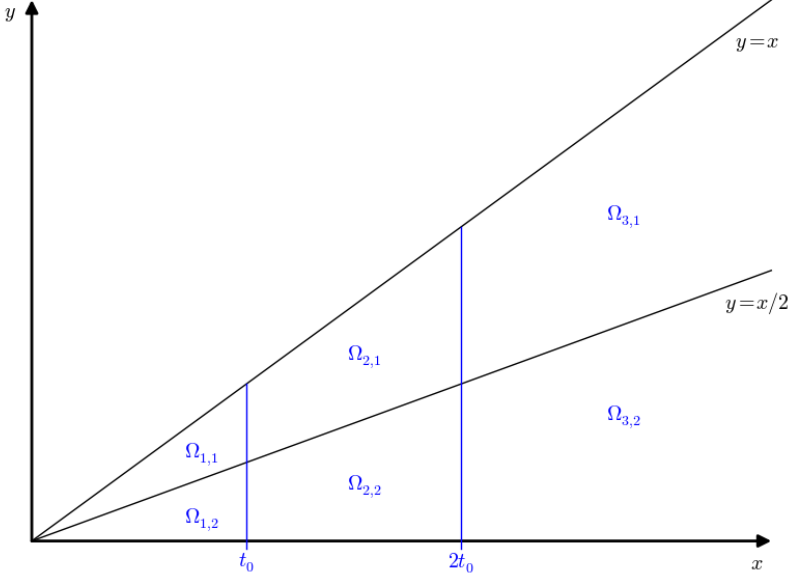


Figure 4

Figure 4 shows the division of the (x, y) -plane into six parts. In the subsequent computations we will refer to the zones $\Omega_{i,j}$, $i, j = 1, 2, 3$.

Step 3.1. Let $x \geq 2t_0$. Now we have to consider two cases, i.e., $0 \leq y \leq x \leq 2y$ ($(x, y) \in \Omega_{3,1}$) and $0 \leq 2y < x$ ($(x, y) \in \Omega_{3,2}$).

Substep 3.1.1. Suppose $(x, y) \in \Omega_{3,1}$, i.e., $\min(w_*(y), w_*(x-y)) = w_*(x-y)$. We consider the function

$$h(x, y) := w_*(x) - w_*(y) - (1-s)w_*(x-y), \quad 0 < y < x.$$

Looking for extreme values of h we have to determine the solutions of $\text{grad } h = 0$. We obtain

$$\begin{aligned} w'_*(x) - (1-s)w'_*(x-y) &= 0, \\ -w'_*(y) + (1-s)w'_*(x-y) &= 0, \end{aligned}$$

which immediately yields $w'_*(x) = w'_*(y)$. Since $w'(t)$ is strictly monotonically decreasing for all $t \geq t_0$ we conclude $x = y$. However, this results in $w'_*(y) = w'_*(x) = 0$ since $w'_*(0) = 0$. Because of $w'_*(t) > 0$ for all $t > 0$ we get a contradiction to $x \geq 2t_0$. Thus, the maximum of $h = h(x, y)$ with respect to $\Omega_{3,1}$ lies on the boundary of the domain, that is, $x = 2t_0$, $y = x$ and $y = \frac{x}{2}$.

Hence we need to check whether the following functions are nonpositive:

$$h(2t_0, y) = w_*(2t_0) - w_*(y) - (1-s)w_*(2t_0 - y) \quad \text{if } t_0 \leq y \leq 2t_0, \quad (4.2)$$

$$h(x, x) = -(1-s)w_*(0) \quad \text{if } 2t_0 \leq x, \quad (4.3)$$

$$h(x, \frac{x}{2}) = w_*(x) - (2-s)w_*(\frac{x}{2}) \quad \text{if } 2t_0 \leq x. \quad (4.4)$$

The function (4.3) is trivially nonpositive for all $s \in (0, 1]$. Considering the function (4.2) and taking account of the mean value theorem we obtain

$$h(2t_0, y) \leq 0 \quad \Longleftrightarrow \quad w'_*(\xi) \leq (1-s) \frac{w_*(2t_0 - y)}{2t_0 - y},$$

where $\xi \in (y, 2t_0)$. A substitution yields

$$w'_*(\xi) \leq (1-s) \frac{w_*(z)}{z} \quad (4.5)$$

with $0 \leq z \leq t_0$ and $\xi \in (2t_0 - z, 2t_0)$. Using the monotonicity of q and of w'_* , see Step 1, we find

$$q(z) w'_*(\xi) = \frac{z}{w_*(z)} w'_*(\xi) \leq \frac{t_0}{w_*(t_0)} w'_*(\xi) \leq \frac{t_0}{w_*(t_0)} w'_*(t_0) \leq p_0 < 1. \quad (4.6)$$

Thus, (4.5) is true with $0 < s \leq 1 - p_0$ and therefore $h(2t_0, y) \leq 0$ on $[t_0, 2t_0]$. It is left to show that the function (4.4) is nonpositive. First of all observe that

$$w'_*(\frac{x}{2}) \leq (1-s) \frac{w_*(\frac{x}{2})}{\frac{x}{2}}, \quad x \geq 2t_0,$$

would imply $h(x, \frac{x}{2}) \leq 0$ (we used the mean value theorem and the fact that $w'_*(t)$ is monotonically decreasing for all $t \geq t_0$). Equivalently we can write

$$p(\frac{t}{2}) = \frac{\frac{x}{2} w'_*(\frac{x}{2})}{w_*(\frac{x}{2})} \leq 1 - s.$$

As above this is true for all $0 < s \leq 1 - p_0$.

Substep 3.1.2. Suppose $(x, y) \in \Omega_{3,2}$, i.e., $\min(w_*(y), w_*(x - y)) = w_*(y)$. Therefore we consider the function

$$h(x, y) := w_*(x) - w_*(x - y) - (1-s)w_*(y), \quad x \geq 2t_0, \quad 0 < y \leq \frac{x}{2}.$$

Considering the equation $\text{grad } h = 0$ we obtain

$$\begin{aligned} w'_*(x) - w'_*(x - y) &= 0, \\ w'_*(x - y) - (1-s)w'_*(y) &= 0. \end{aligned}$$

Taking care of the first equation only we get $w'_*(x) = w'_*(x - y)$. Due to the assumptions we also know that $x \geq x - y \geq t_0$. From the monotonicity of w' on $[t_0, \infty)$ we conclude that $y = 0$ and then, by means of the second equation, $w'_*(x) = 0$. Again it follows $x = 0$ which is in contradiction with

the assumption $x \geq 2t_0$. Thus, $h = h(x, y)$ attains it's maximum on the boundary of $\Omega_{3,2}$. We have to consider

$$h(2t_0, y) = w_*(2t_0) - w_*(2t_0 - y) - (1 - s)w_*(y) \quad \text{if } 0 \leq y < t_0, \quad (4.7)$$

$$h(x, 0) = -(1 - s)w_*(0) \quad \text{if } 2t_0 \leq x, \quad (4.8)$$

$$h(x, \frac{x}{2}) = w_*(x) - (2 - s)w_*(\frac{x}{2}) \quad \text{if } 2t_0 \leq x. \quad (4.9)$$

The functions in (4.8) and (4.9) were already considered in Substep 3.1.1. It remains to show that the function (4.7) is nonpositive. Applying the same arguments as in Substep 3.1.1 we get

$$h(2t_0, y) = yw'_*(\xi) - (1 - s)w_*(y) \leq 0, \quad 2t_0 - y < \xi < 2t_0.$$

This would follow from

$$\frac{y}{w_*(y)}w'_*(\xi) \leq 1 - s,$$

if the latter inequality would be true for all $y \in [0, t_0)$ and all $\xi \in (2t_0 - y, 2t_0)$. Arguing as in (4.6) we find

$$\frac{y}{w_*(y)}w'_*(\xi) \leq \frac{t_0}{w_*(t_0)}w'_*(\xi) \leq \frac{t_0}{w_*(t_0)}w'_*(t_0) \leq p_0,$$

which shows that $h(2t_0, y)$ is nonpositive if $0 < s \leq 1 - p_0$.

Step 3.2. Let $t_0 < x < 2t_0$. We cannot apply the strategy of Step 1 anymore since $w''_* = w''_*(y)$ changes sign in the considered interval for y .

Substep 3.2.1. Suppose $(x, y) \in \Omega_{2,1}$, i.e., $\min(w_*(y), w_*(x - y)) = w_*(x - y)$. Therefore we consider the function

$$h(x, y) := w_*(x) - w_*(y) - (1 - s)w_*(x - y), \quad t_0 < x < 2t_0, \quad \frac{x}{2} < y < x,$$

which is nonpositive if and only if

$$\frac{w_*(x) - w_*(y)}{w_*(x - y)} \stackrel{[z=x-y]}{=} \frac{w_*(z + y) - w_*(y)}{w_*(z)} \leq 1 - s. \quad (4.10)$$

By the mean value theorem we obtain

$$\frac{w_*(z + y) - w_*(y)}{w_*(z)} = \frac{z}{w_*(z)}w'_*(\xi_{y,z}),$$

where $z \in [0, t_0)$ and $\xi_{y,z} \in (y, z + y)$. We know that $w''_* = w''_*(t)$ vanishes if and only if $t = t_0$. Thus, $w'_* = w'_*(t)$ attains it's global maximum at $t = t_0$. The monotonicity of the function q leads to

$$\frac{z}{w_*(z)}w'_*(\xi_{y,z}) \leq \frac{z}{w_*(z)}w'_*(t_0) \leq \frac{t_0}{w_*(t_0)}w'_*(t_0) \leq p_0.$$

Hence $h(x, y) \leq 0$ if $0 < s \leq 1 - p_0$.

Substep 3.2.2. Suppose $(x, y) \in \Omega_{2,2}$, i.e., $\min(w_*(y), w_*(x - y)) = w_*(y)$. Therefore we consider the function

$$h(x, y) := w_*(x) - w_*(x - y) - (1 - s)w_*(y), \quad t_0 < x < 2t_0, \quad 0 < y < \frac{x}{2},$$

which is nonpositive if and only if

$$\frac{w_*(x) - w_*(x-y)}{w_*(y)} = \frac{y}{w_*(y)} w'_*(\xi_{x,y}) \leq 1 - s \quad (4.11)$$

with $\xi_{x,y} \in (x-y, x)$. By the same arguments as in Substep 3.2.1 we get $h(x, y) \leq 0$ if $0 < s \leq 1 - p_0$.

Step 3.3. Let $0 \leq x \leq t_0$. Remark that all the substeps are treated as in Step 1. with the difference in the interval with respect to the x -variable.

Substep 3.3.1. Suppose $(x, y) \in \Omega_{1,1}$, i.e., $\min(w_*(y), w_*(x-y)) = w_*(x-y)$. Therefore we consider the function

$$h(x, y) := w_*(x) - w_*(y) - (1-s)w_*(x-y), \quad 0 < x < t_0, \quad \frac{x}{2} < y < x.$$

Analogously to Step 3.1.1, equation $\text{grad } h = 0$ yields $x = y = 0$. Consequently the maximal value of h has to be located at the boundary. It remains to consider the functions $h(x, x)$ and $h(x, \frac{x}{2})$. The first one is trivial. Concerning the second one we observe

$$h(x, y) \leq 0 \quad \Longleftrightarrow \quad \frac{x}{2} \frac{w'_*(\xi)}{w_*(\frac{x}{2})} \leq 1 - s.$$

Employing the monotonicity properties of p and q we get

$$\frac{x}{2} \frac{w'_*(\xi)}{w_*(\frac{x}{2})} \leq t_0 \frac{w'_*(\xi)}{w_*(t_0)} \leq p(t_0) \leq p_0.$$

Hence, $h(x, y) \leq 0$ if $0 < s \leq 1 - p_0$.

Step 3.3.2. Suppose $(x, y) \in \Omega_{1,2}$, i.e., $\min(w_*(y), w_*(x-y)) = w_*(y)$. Therefore we consider the function

$$h(x, y) := w_*(x) - w_*(x-y) - (1-s)w_*(y), \quad 0 < x < t_0, \quad 0 < y < \frac{t_0}{2}.$$

Again the equation $\text{grad } h = 0$ yields $x = y = 0$. This justifies to study h on $\partial\Omega_{1,2}$ only. The behavior of $h(x, \frac{x}{2})$ we have already investigated in Substep 3.3.1. Since the function $h(x, 0) \leq 0$ for all $s \in (0, 1]$ we are done. \square

Remark 4.5. Our proof yields that any s , $0 < s \leq 1 - p_0$ will do the job. Here $p_0 = \sup_{t>0} p(t) \in (0, 410247, 0, 410248)$.

Based on Lemma 4.4 we may follow the arguments in the proof of Theorem 2.15 and Corollary 2.17.

Theorem 4.6. Let $1 \leq p_1, p_2, q \leq \infty$. Define p by $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$. Assume that $f \in \mathcal{UM}^{p_1, q}(\mathbb{R}^n)$ and $g \in \mathcal{UM}^{p_2, q}(\mathbb{R}^n)$. Then $f \cdot g \in \mathcal{UM}^{p, q}(\mathbb{R}^n)$ and it holds

$$\|f \cdot g\|_{\mathcal{UM}^{p, q}} \leq C \|f\|_{\mathcal{UM}^{p_1, q}} \|g\|_{\mathcal{UM}^{p_2, q}}$$

with a positive constant C which only depends on the choice of the frequency-uniform decomposition, the dimension n and the parameter q .

In particular, the modulation space $\mathcal{UM}^{p, q}(\mathbb{R}^n)$ is an algebra under pointwise multiplication.

Proof. Step 1. We can basically follow the proof of Theorem 2.15. Therefore we will restrict ourselves to very few comments. The first comment concerns the constant c_3 . Let s denote the positive constant in Lemma 4.4. This time we define

$$c_3 := \left(\sum_{m \in \mathbb{Z}^n} e^{-s w_*(|m|) q'} \right)^{\frac{1}{q'}} < \infty, \quad (4.12)$$

see (2.4). Also the constant c_4 , see (2.5), requires some changes. The uniform boundedness and the positivity of w'_* and the mean-value theorem imply the existence of some ξ such that

$$e^{w_*(|j|) - w_*(|t-j|)} = e^{w'_*(\xi)(|j| - |j-t|)} \leq e^{\|w'_*\|_{L^\infty} |t|}.$$

Hence,

$$\max_{\substack{t \in \mathbb{Z}^n, \\ -3 < t_i < 3, \\ i=1, \dots, n}} \sup_{j \in \mathbb{Z}^n} e^{w_*(|j|) - w_*(|t-j|)} \leq e^{\|w'_*\|_{L^\infty} 2\sqrt{n}} =: c_4 < \infty. \quad (4.13)$$

All the remaining arguments do not change.

Step 2. We prove the algebra property of $\mathcal{UM}^{p,q}(\mathbb{R}^n)$. An application of Lemma 2.9 yields the continuous embedding

$$\mathcal{UM}^{p_1,q}(\mathbb{R}^n) \hookrightarrow \mathcal{UM}^{p_2,q}(\mathbb{R}^n), \quad 1 \leq p_1 \leq p_2 \leq \infty.$$

Now we apply Step 1 with $p_1 = p_2 = 2p$ and use the embedding $\mathcal{UM}^{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{UM}^{p_j,q}(\mathbb{R}^n)$, $j = 1, 2$. The proof is complete. \square

Now the same steps as in Section 3 yield a non-analytic superposition result on the modulation spaces $\mathcal{UM}^{p,q}(\mathbb{R}^n)$. First of all we need to show the subalgebra property. Therefore recall the decomposition of the phase space in $(2^n + 1)$ parts as in Section 3.

Proposition 4.7. *Let $1 \leq p, q \leq \infty$ and $N \in \mathbb{N}$. Suppose that $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is fixed with $\epsilon_j \in \{0, 1\}$, $j = 1, \dots, n$. Let s be as in Lemma 4.4 and $R \geq 2$. The spaces*

$$\mathcal{UM}^{p,q}(\epsilon, R) := \{f \in \mathcal{UM}^{p,q}(\mathbb{R}^n) : \text{supp } \mathcal{F}(f) \subset P_R(\epsilon)\}$$

are subalgebras of $\mathcal{UM}^{p,q}$. Furthermore, it holds

$$\|fg\|_{\mathcal{UM}^{p,q}} \leq G_{R,N} \|f\|_{\mathcal{UM}^{p,q}} \|g\|_{\mathcal{UM}^{p,q}} \quad (4.14)$$

for all $f, g \in \mathcal{UM}^{p,q}(\epsilon, R)$. The constant $G_{R,N}$ can be specified by

$$G_{R,N} = C_N R^{-N},$$

where the constant $C_N > 0$ depends on n, p, q and N , but not on f, g and R .

Proof. We proceed as in proof of Proposition 3.1 (using also the notation introduced there). Hence, it will be enough to discuss the changes caused by

the replacement of $|k|^{1/s}$ by $w_*(|k|)$. To estimate the counterparts of $S_{1,t,k}$ and $S_{2,t,k}$ we need to estimate the quantity

$$\tilde{G}_R := \left(\sum_{\substack{l \in \mathbb{Z}^n: l, t - (l-k) \in P_R^*(0, \dots, 0), \\ |l| \leq |l-k|}} e^{-sq'w_*(|l|)} \right)^{1/q'}.$$

Here we will use that for any natural number M there exists a constant c_M such that

$$\sup_{l \in \mathbb{Z}^n} e^{-sq'w_*(|l|)} (1 + |l|)^M =: c_M < \infty.$$

By definition of $P_R^*(0, \dots, 0)$ it follows

$$\begin{aligned} \tilde{G}_R &\leq c_M \left(\sum_{l \in \mathbb{Z}^n: \|l\|_\infty > R-1} (1 + |l|)^{-Mq'} \right)^{1/q'} \\ &\leq c_M \left(\int_{\|x\|_\infty > R-2} (1 + |x|)^{-Mq'} dx \right)^{1/q'} \\ &\leq c_M \left(\int_{|x| > R-2} (1 + |x|)^{-Mq'} dx \right)^{1/q'}. \end{aligned}$$

Clearly, if $Mq' > n$ we obtain

$$\begin{aligned} \int_{|x| > R-2} (1 + |x|)^{-Mq'} dx &= 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \int_{R-2}^\infty r^{n-1} (1 + r)^{-Mq'} dr \\ &\leq 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \int_{R-1}^\infty r^{n-1-Mq'} dr \\ &= 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{(R-1)^{n-Mq'}}{Mq' - n}. \end{aligned}$$

Altogether this results in the estimate

$$\tilde{G}_R \leq C (R-1)^{\frac{n}{q'} - M},$$

where $C = C(n, q, M)$ depends on n, q, M but not on f, g and R . For given $N \in \mathbb{N}$, by choosing $M - \frac{n}{q'} \geq N$, we can always guarantee an estimate of \tilde{G}_R by a constants times R^{-N} for all $R \geq 2$. Taking into account the other constants showing up during the proof of Proposition 3.1 (but see also the proofs of Theorem 2.15 and Corollary 2.17) we finally get (4.14). \square

Lemma 4.8. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then, for any $\vartheta > 1$ and for any $N \in \mathbb{N}$ there exist positive constants b, c such that*

$$\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}} \leq c \|u\|_{\mathcal{UM}^{p,q}} \begin{cases} e^{\vartheta w_*(b \|u\|_{\mathcal{UM}^{p,q}}^{1+\frac{1}{N}})} & \text{if } \|u\|_{\mathcal{UM}^{p,q}} > 1, \\ 1 & \text{if } \|u\|_{\mathcal{UM}^{p,q}} \leq 1 \end{cases}$$

holds for all $u \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$. In addition, the constant b can be chosen independent of ϑ and N .

Proof. We follow the proof of Lemma 3.6.

Step 1. Thus, we let $u \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$ satisfying $\text{supp } \mathcal{F}(u) \subset P_R$. Again we employ the Taylor expansion of e^{iu} to get the norm estimate

$$\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}} \leq \left\| \sum_{l=1}^r \frac{(iu)^l}{l!} \right\|_{\mathcal{UM}^{p,q}} + \left\| \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!} \right\|_{\mathcal{UM}^{p,q}} =: S_1 + S_2.$$

After some computations analogously to the proof of Lemma 3.6, see in particular (3.3), we obtain

$$S_2 \leq C_2 \|u\|_{\mathcal{UM}^{p,q}}, \quad C_2 := \left(C_1 e, \frac{3}{3-e} \right),$$

where this time C_1 means the algebra constant with respect to $\mathcal{UM}^{p,q}(\mathbb{R}^n)$, see Theorem 4.6. Concerning S_1 we conclude

$$S_1 \leq \left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{w_*(|k|)q} \|\square_k(e^{iu} - 1)\|_{L^p}^q \right)^{\frac{1}{q}} + S_2.$$

Because of

$$\left(\sum_{\substack{k \in \mathbb{Z}^n, \\ -Rr-1 < k_i < Rr+1, \\ i=1, \dots, n}} e^{w_*(|k|)q} \right)^{\frac{1}{q}} \leq c_1 e^{w_*(\sqrt{n}(Rr+1))} (\sqrt{n}(Rr+1))^{n/q}$$

(here c_1 depends on n only) we find

$$S_1 \leq (2 C_3 C_4 C_5 e^{\vartheta w_*(\sqrt{n}(Rr+1))} + C_2) \|u\|_{\mathcal{UM}^{p,q}},$$

where C_2 has the above meaning and C_3 has the original meaning from the proof of Lemma 3.6. The definitions of other two constants have to be modified. C_5 is defined as

$$C_5 := \left(\sum_{k \in \mathbb{Z}^n} e^{-w_*(|k|)q'} \right)^{\frac{1}{q'}} < \infty,$$

whereas C_4 is now given by

$$C_4 := \sup_{r \in \mathbb{N}} \sup_{R \geq 2} c_1 n^{\frac{n}{2q}} (Rr+1)^{n/q} e^{-(\vartheta-1)w_*(\sqrt{n}(Rr+1))} < \infty.$$

Recall that we had chosen r such that

$$3 C_1 \|u\|_{\mathcal{UM}^{p,q}} \leq r \leq 3 C_1 \|u\|_{\mathcal{UM}^{p,q}} + 1,$$

see (3.2). By means of this we can rewrite our estimate of $\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}}$ and get

$$\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}} \leq c_0 \|u\|_{\mathcal{UM}^{p,q}} \left(1 + e^{\vartheta w_*(b_0 R \|u\|_{\mathcal{UM}^{p,q}})} \right), \quad (4.15)$$

compare with (3.5), valid for all $u \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$ satisfying $\text{supp } \mathcal{F}(u) \subset P_R$ and with positive constants b_0, c_0 depending on n, p and q but independent of u, r and R .

Step 2. Now we consider the general case. Let $u \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$. Again we

decompose u in the phase space according to Lemma 3.6. Let $G_{R,N}$ be the constant in (4.14) and let C_6 be as in proof of Lemma 3.6. Then it follows

$$\begin{aligned} \|e^{iu_{j_k}} - 1\|_{\mathcal{UM}^{p,q}} &= \left\| \sum_{l=1}^{\infty} \frac{(iu_{j_k})^l}{l!} \right\|_{\mathcal{UM}^{p,q}} \leq \frac{1}{G_{R,N}} \left(e^{G_{R,N} \|u_{j_k}\|_{\mathcal{UM}^{p,q}}} - 1 \right) \\ &\leq \frac{1}{G_{R,N}} \left(e^{G_{R,N} C_6 \|u\|_{\mathcal{UM}^{p,q}}} - 1 \right), \end{aligned} \quad (4.16)$$

see (3.9), as well as

$$\|e^{iu_0} - 1\|_{\mathcal{UM}^{p,q}} \leq c_0 C_6 \|u\|_{\mathcal{UM}^{p,q}} \left(1 + e^{\vartheta w_*(b_0 R \|u\|_{\mathcal{UM}^{p,q}})} \right), \quad (4.17)$$

see (3.10).

Substep 2.1. Let $\|u\|_{\mathcal{UM}^{p,q}} \leq 1$. We choose $R = 3$. As in proof of Lemma 3.6 we conclude

$$\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}} \leq c_2 \|u\|_{\mathcal{UM}^{p,q}}, \quad (4.18)$$

where c_2 does not depend on u .

Substep 2.2. Let $\|u\|_{\mathcal{UM}^{p,q}} > 1$. We know that the algebra constant $G_{R,N}$ in (4.16) is a function of R , more exactly,

$$G_{R,N} = \tilde{C}_N R^{-N}.$$

Taking into account that

- $G_{R,N}$ is strictly decreasing and positive and
- $\lim_{R \rightarrow \infty} G_{R,N} = 0$

we may choose $R > 2$ according to

$$\left(\frac{2}{R} \right)^N = \frac{G_{R,N}}{G_{2,N}} = \|u\|_{\mathcal{UM}^{p,q}}^{-1}.$$

This means in particular

$$G_{R,N} \|u\|_{\mathcal{UM}^{p,q}} = \tilde{C}_N 2^{-N} \quad \text{and} \quad R = 2 \|u\|_{\mathcal{UM}^{p,q}}^{1/N}.$$

Now (4.16) and (4.17) result in

$$\begin{aligned} &\|e^{iu} - 1\|_{\mathcal{UM}^{p,q}} \\ &\leq c_3 \max_{\alpha \in \{0,1\}, \beta \in \{0, \dots, 2^n\}} \left(c_0 C_6 \|u\|_{\mathcal{UM}^{p,q}} \left(1 + e^{\vartheta w_*(b_0 R \|u\|_{\mathcal{UM}^{p,q}})} \right) \right)^\alpha \\ &\quad \times \left(\frac{e^{G_{R,N} C_6 \|u\|_{\mathcal{UM}^{p,q}}} - 1}{G_{R,N}} \right)^\beta \\ &\leq c_4 \max_{\alpha \in \{0,1\}, \beta \in \{0, \dots, 2^n\}} \left(\|u\|_{\mathcal{UM}^{p,q}} \left(1 + e^{\vartheta w_*(b_0 2 \|u\|_{\mathcal{UM}^{p,q}}^{1+\frac{1}{N}})} \right) \right)^\alpha \|u\|_{\mathcal{UM}^{p,q}}^\beta \\ &\quad \times \left(\frac{e^{G_{2,N} C_6} - 1}{G_{2,N}} \right)^\beta \\ &\leq c_5 \|u\|_{\mathcal{UM}^{p,q}} \left(1 + e^{\vartheta w_*(b_0 2 \|u\|_{\mathcal{UM}^{p,q}}^{1+\frac{1}{N}})} \right) \end{aligned}$$

with a constant c_5 depending on ϑ and N but independent of u . \square

Lemma 4.9. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Assume $u \in \mathcal{UM}^{p,q}(\mathbb{R}^n)$ to be fixed and define a function $g : \mathbb{R} \mapsto \mathcal{UM}^{p,q}(\mathbb{R}^n)$ by $g(\xi) = e^{iu(x)\xi} - 1$. Then the function g is continuous.*

Proof. This lemma can be proved in the same way as Lemma 3.8. \square

Theorem 4.10. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\varepsilon > 0$ and $\vartheta > 1$. Let μ be a complex measure on \mathbb{R} such that*

$$L_1(\lambda) := \int_{\mathbb{R}} e^{\vartheta \log(\langle \lambda |\xi|^{1+\varepsilon} \rangle_*) \log \log(\langle \lambda |\xi|^{1+\varepsilon} \rangle_*)} d|\mu|(\xi) < \infty \quad (4.19)$$

for any $\lambda > 0$ and such that $\mu(\mathbb{R}) = 0$.

Furthermore, assume that the function f is the inverse Fourier transform of μ . Then $f \in C^\infty$ and the composition operator $T_f : u \mapsto f \circ u$ maps $\mathcal{UM}^{p,q}$ into $\mathcal{UM}^{p,q}$.

Proof. We exactly follow the proof of Theorem 3.9. Using equation (4.19), Lemma 4.9 and Lemma 4.8 complete the proof. \square

Note the following conclusion.

Corollary 4.11. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Let μ be a complex measure on \mathbb{R} with the corresponding bounded density function g , i.e., $d\mu(\xi) = g(\xi) d\xi$. Suppose that*

$$\lim_{|\xi| \rightarrow \infty} \frac{\log(\langle |\xi| \rangle_*) \log \log(\langle |\xi| \rangle_*)}{\log |g(\xi)|} = 0$$

and $\int_{\mathbb{R}} d\mu(\xi) = \int_{\mathbb{R}} g(\xi) d\xi = 0$. Assume the function f to be the inverse Fourier transform of g . Then $f \in C^\infty$ and the composition operator $T_f : u \mapsto f \circ u$ maps $\mathcal{UM}^{p,q}$ into $\mathcal{UM}^{p,q}$.

Proof. Without loss of generality let $0 < \varepsilon < 1$. Making use of the elementary inequality

$$w_*(|\xi|^{1+\varepsilon}) \leq (1 + \varepsilon) \log(\langle |\xi| \rangle_*) \left(\varepsilon + \log \log(\langle |\xi| \rangle_*) \right)$$

we can follow the proof of Corollary 3.11. \square

One Example

By \mathcal{X} we denote the characteristic function of the interval $[0, 1)$. Then we define the special function

$$\text{up}(x) := \lim_{r \rightarrow \infty} \left(\mathcal{X} * (2 \mathcal{X}(2 \cdot) * \dots * (2^r \mathcal{X}(2^r \cdot)) \right)(x), \quad x \in \mathbb{R},$$

originally introduced by the brothers Rvatchev [19]. A good source represents the survey article [20]. This function has a number of nice properties, e.g.,

$$\text{up}'(x) = 2\text{up}(2x + 1) - 2\text{up}(2x - 1), \quad x \in \mathbb{R}.$$

Here we need that $\text{up} \in C_0^\infty(\mathbb{R})$, $\text{up} \geq 0$ on \mathbb{R} , $\text{up}(x) > 0$ if $0 < x < 2$, and

$$\mathcal{F}\text{up}(\xi) = \frac{e^{-i\xi}}{\sqrt{2\pi}} \prod_{j=1}^{\infty} \frac{\sin 2^{-j}\xi}{2^{-j}\xi}, \quad \xi \in \mathbb{R}.$$

It is an exercise to prove the decay estimate

$$\begin{aligned} |\mathcal{F}\text{up}(\xi)| &\leq \frac{1}{\sqrt{2\pi}} |\xi|^{1-(\log_2 |\xi|)/2} \\ &= \frac{1}{\sqrt{2\pi}} e^{(1-(\log_2 |\xi|)/2) \log |\xi|} \quad \text{if } |\xi| \geq 1. \end{aligned}$$

Corollary 4.12. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then the composition operator $T_{\text{up}} : u \mapsto \text{up} \circ u$ maps $\mathcal{UM}^{p,q}$ into $\mathcal{UM}^{p,q}$.*

5. Concluding Remarks and Open Questions

- The main issue of this paper has been to explain non-analytic superposition on modulation spaces with subexponential weights. Up to now we have got results in two special situations, namely in Section 3 we studied Gevrey-modulation spaces, in Section 4 we studied $\mathcal{UM}^{p,q}$. In a forthcoming project we would like to extend the theory of non-analytic superpositions to more general modulation spaces of ultra-differentiable type.
- In an other forthcoming project we will study superposition operators on modulation spaces of Sobolev type, too. Here two properties seem to be essential for this issue.

Proposition 5.1. ([22]) *Let $1 \leq p, q \leq \infty$. If $s > \frac{n}{q'}$, where q' is the conjugate to q , then $M_{p,q}^s(\mathbb{R}^n)$ is an algebra with respect to pointwise multiplication.*

The condition $s > \frac{n}{q'}$ also implies $M_{p,q}^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, see [22]. In some sense we believe that for the study of superposition operators on modulation spaces of Sobolev type this condition is a threshold for inner functions in a similar way as for the classical Sobolev spaces itself, see Bourdaud [3], [18, 5.2.4] or Bourdaud, S. [5].

- We are able to apply the superposition results to handle nonlinear partial differential equations. By the concepts of [4] we can investigate the solutions of (1.3) and (1.4) which were introduced in Section 1. We expect local (in time) existence results for data belonging to suitable Gevrey-modulation spaces. We can also apply the modulation spaces of Section 4 if the coefficient $a = a(t)$ in (1.4) has a suitable modulus of continuity behavior. However within the scope of this paper only the relevant tools are provided. In future work we will study the existence of locally (in time) and globally (in time) solutions.

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